Taylor Table To Derive a Generalized $4^{th}$ to $6^{th}$ Order Compact Pade Scheme

T. H. Pulliam

NASA Ames
1. The generalized form of the equation is given by
\[
\left( \frac{\partial u}{\partial x} \right)_{j-1} + \alpha \left( \frac{\partial u}{\partial x} \right)_j + \left( \frac{\partial u}{\partial x} \right)_{j+1} - \frac{A}{2\Delta x} (-u_{j-1} + u_{j+1}) - \frac{B}{4\Delta x} (-u_{j-2} + u_{j+2}) = \varepsilon r_t
\]

1. The equation is written on terms of the single free coefficients \( \alpha, A, B \) which must be determined using the Taylor Table approach as outlined below.

2. One goal is to find the value of \( \alpha, A, B \) which results in a 6\(^{th}\) Order Scheme

3. We can also define a class of 4\(^{th}\) schemes where \( \alpha \) is a free parameter and \( A, B \) are functions of \( \alpha \)
<table>
<thead>
<tr>
<th>$\Delta x \cdot \left( \frac{\partial u}{\partial x} \right)_{j-1}$</th>
<th>$u_j$</th>
<th>$\Delta x \cdot \left( \frac{\partial u}{\partial x} \right)_{j}$</th>
<th>$\Delta x \cdot \alpha \left( \frac{\partial u}{\partial x} \right)_{j}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta x \cdot \left( \frac{\partial u}{\partial x} \right)_{j+1}$</td>
<td>$\frac{A}{2} u_{j-2}$</td>
<td>$\frac{A}{2} (-1) \frac{1}{1!}$</td>
<td>$\frac{A}{2} (-1)^2 \frac{1}{2!}$</td>
</tr>
<tr>
<td>$- \frac{B}{4} u_{j+2}$</td>
<td>$- \frac{B}{4} (-2) \frac{1}{1!}$</td>
<td>$- \frac{B}{4} (-2)^2 \frac{1}{2!}$</td>
<td>$- \frac{B}{4} (-2)^3 \frac{1}{3!}$</td>
</tr>
</tbody>
</table>

$\Delta x \cdot \left( \frac{\partial u}{\partial x} \right)_{j-1}$

$\Delta x \cdot \alpha \left( \frac{\partial u}{\partial x} \right)_{j}$
1. For Consistency and at least 4\textsuperscript{th} Order Accuracy, the first five columns are set to zero.

2. Note because of the skew symmetry of the original equation the odd numbers columns sum exact to 0

3. For 6\textsuperscript{th} Order Accuracy we need

\[
\alpha + 2 - A - B = 0, \quad 1 - \frac{A}{6} - \frac{2B}{3} = 0, \quad 2 - \frac{A}{5} - \frac{16B}{5} = 0
\] (1)

1. Solving we have \( \alpha = 3, \ A = \frac{14}{3}, \) and \( B = \frac{1}{3} \)

2. Then \( er_t = -\frac{1}{180} \Delta x^6 \left( \frac{\partial^7 u}{\partial x^7} \right)_j \)
1. Instead of requiring a $6^{th}$ order scheme relax the conditions to allow the sixth column to be non-zero and find $A, B$ as a function of $\alpha$

2. Solving the first two relations for $A, B$ we have $A = \frac{4+2\alpha}{3}$ and $B = \frac{4-\alpha}{3}$

3. For $\alpha = 3$: the above 5 point $6^{th}$ Order Scheme

4. For $\alpha = 4$: the 3 point $4^{th}$ Order Scheme

5. For $\alpha \neq 3$: a class of $4^{th}$ Order Schemes different characteristics.

6. In general,

   \[ er_t = \Delta x^4 \frac{1}{10} \left( 1 - \frac{\alpha}{3} \right) \left( \frac{\partial^5 u}{\partial x^5} \right)_j + \Delta x^6 \frac{1}{1260} \left( 8 - 5\alpha \right) \left( \frac{\partial^7 u}{\partial x^7} \right)_j \]

7. See Note on Modified Wave Number for General Pade Schemes