Euler Equation - Wave Equation Connection

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1. The One-dimensional (1-D) Wave Equation

\[ \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \]  

with \( a \) the wave speed.

2. Is a good representative equation for the Euler Equations

3. First part of the course we will use the 1-D Wave Equation to derive and analyze various aspects of accuracy, stability and efficiency

4. What motivates this model Equation?
1. The Euler Equations are

\[
\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = 0
\]  

(2)

\[
Q = \begin{bmatrix}
\rho \\
\rho u \\
e
\end{bmatrix}, \quad E = \begin{bmatrix}
\rho u \\
\rho u^2 + p \\
u(e + p)
\end{bmatrix}
\]  

(3)

Equation of state

\[
p = (\gamma - 1) \left( e - \frac{1}{2} \rho (u^2) \right)
\]  

(4)

where \( \gamma \) is the ratio of specific heats, generally taken as 1.4.
Quasi-Linear Form

1. First we re-write the Euler Equations, Eq. 2, in chain rule form (Quasi-Linear)

2. Let \( \frac{\partial E}{\partial x} = \left( \frac{\partial E}{\partial Q} \right) \frac{\partial Q}{\partial x} \), where \( \frac{\partial E}{\partial Q} \) needs to be defined since \( E \) and \( Q \) are vectors.

3. The term \( \frac{\partial E}{\partial Q} \) is a tensor, actually a Matrix defined as the Jacobian of the Flux Vector \( E \) with respect to \( Q \).

4. Eq.2 can be rewritten as (A defined below)

\[
\frac{\partial Q}{\partial t} + A \frac{\partial Q}{\partial x} = 0
\]  

(5)
Generalized Forms

1. Redefine \( Q \) and \( E \) in terms of Independent Variables \( q_1, q_2, q_3 \) as

\[
Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}
\]

\[
E = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \frac{q_2}{q_1} + (\gamma - 1) \left( q_3 - \frac{1}{2} \frac{q_2^2}{q_1} \right) \\ \frac{q_2}{q_1} \left( q_3 + (\gamma - 1) \left( q_3 - \frac{1}{2} \frac{q_2^2}{q_1} \right) \right) \end{bmatrix}
\]
1. The definition of the Jacobian $A = \frac{\partial E}{\partial Q}$,

$$A = \begin{bmatrix}
\frac{\partial e_1}{\partial q_1} & \frac{\partial e_1}{\partial q_2} & \frac{\partial e_1}{\partial q_3} \\
\frac{\partial e_2}{\partial q_1} & \frac{\partial e_2}{\partial q_2} & \frac{\partial e_2}{\partial q_3} \\
\frac{\partial e_3}{\partial q_1} & \frac{\partial e_3}{\partial q_2} & \frac{\partial e_3}{\partial q_3}
\end{bmatrix}$$

$$A = \begin{bmatrix}
0 & 1 & 0 \\
\frac{\gamma - 3}{2}u^2 & (3 - \gamma)u & \gamma - 1 \\
-\frac{\gamma eu}{\rho} + (\gamma - 1)u^3 & \frac{\gamma e}{\rho} - \frac{3(\gamma - 1)u^2}{2} & \gamma u
\end{bmatrix}$$
Linear Diagonalized Form of Euler Equations

1. Freeze the Jacobian Matrix $A$ at a reference state $A_0$

2. This can be justified by small perturbation theory, asymptotic analysis, etc.

3. We now have

$$\frac{\partial Q}{\partial t} + A_0 \frac{\partial Q}{\partial x} = 0$$  \hspace{1cm} (6)

4. The matrix $A$ (and the corresponding $A_0$) has a complete set of eigenvectors and eigenvalues.
1. Let $A = X\Lambda X^{-1}$ and conversely $\Lambda = X^{-1}AX$

(a) $X$ is the $3 \times 3$ eigenvector matrix of $A$

(b) $\Lambda$ is the diagonal eigenvalue matrix with elements, $\lambda_1, \lambda_2, \lambda_3$.

(c) For the Euler Equations, $\lambda_1 = u, \lambda_2 = u + c$, and $\lambda_3 = u - c$ with $c = \sqrt{\frac{\gamma p}{\rho}}$, the speed of sound.
Diagonalization of Euler Equations

1. Using the Eigensystem of $A_0$ we can transform Eq.6 to

$$X_0^{-1} \left[ \frac{\partial Q}{\partial t} + A_0 X_0 X_0^{-1} \frac{\partial Q}{\partial x} \right] = 0$$

$$\frac{\partial \left[ X_0^{-1} Q \right]}{\partial t} + \left[ X_0^{-1} A_0 X_0 \right] \frac{\partial \left[ X_0^{-1} Q \right]}{\partial x} = 0$$

$$\frac{\partial W}{\partial t} + \Lambda_0 \frac{\partial W}{\partial x} = 0$$

(7)

with $W = X_0^{-1} Q$
Characteristics Form of Euler Equations

1. The Equations in Characteristic Form are uncoupled

\[ \frac{\partial w_i}{\partial t} + \lambda_{0_i} \frac{\partial w_i}{\partial x} = 0 \] (8)

for \( i = 1, 2, 3 \)

2. So for each \( i \), we have the wave equation, Eq.1, where \( u = w_i \) and \( a = \lambda_{0_i} \)

3. Therefore, any process, analysis, stability, etc, results applied to the wave equation holds for each characteristic equation of \( w_i \)
1. To Complete the process

(a) Transform back to physical variables \( Q = X_0 W \)

(b) \( X_0 \) is a constant matrix (it is made up of elements at the frozen state and therefore not a function of \( x, t \))

(c) The resulting \( Q \) is just linear combinations of the \( w_i \) and any results applied to \( w_i \) also apply to \( q_i \).

(d) For example, if any of the \( w_i \) are divergent (unstable, going to infinity, inaccurate, etc), the \( q_i \) behave consistently with the \( w_i \)
CONCLUSIONS

1. The wave equation Eq.1:

\[ \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \]  

is an appropriate model equation for the Euler Equations

2. GET USE TO SEEING IT FOR THE NEXT FEW WEEKS!!