A High Resolution Compact Scheme for Compressible Flows Involving Shocks and Turbulence

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NASA AMS Seminar
March 15, 2018
1. Introduction
2. Overview of the WCNS methodology
3. The WCHR scheme
   - Explicit interpolation
   - Explicit-compact interpolation
   - Nonlinear weights
   - Derivative scheme
   - Approximate dispersion relation
   - Extension to Euler equations
4. Results
   - 1D test problems
   - 2D test problems
   - 3D test problems
5. Implementation details
6. Conclusions
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Many practical problems involve shocks and turbulence
Need minimally dissipative scheme to capture turbulence
But enough to stabilize solutions at shocks and avoid spurious oscillations
Conflicting requirements causes high sensitivity to numerical schemes

Outline

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Numerical method

- Consider a scalar conservation law in 1D

\[ \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \]

- Semi-discretize this equation on a grid with \( N \) points in the strong form

\[ \frac{\partial u_j}{\partial t} + \left. \frac{\partial f(u)}{\partial x} \right|_j = 0 \]

- Need a discrete approximation of the flux derivative

\[ \left. \frac{\partial f(u)}{\partial x} \right|_j \]
Overview of the WCNS methodology

Weighted compact nonlinear scheme (WCNS) \(^3\ 4\)

\[
\tilde{u}_j - \frac{3}{2} \quad \tilde{u}_j - \frac{1}{2} \quad \tilde{u}_j + \frac{1}{2} \quad \tilde{u}_j + \frac{3}{2} \quad \tilde{u}_j + \frac{5}{2}
\]

\[
u_{j-2} \quad \nu_{j-1} \quad \nu_j \quad \nu_{j+1} \quad \nu_{j+2} \quad \nu_{j+3}
\]


Overview of the WCNS methodology

Weighted compact nonlinear scheme (WCNS) \(^3\) \(^4\)

\[ \tilde{u}_j - \frac{3}{2} \quad \tilde{u}_j - \frac{1}{2} \quad \tilde{u}_j + \frac{1}{2} \quad \tilde{u}_j + \frac{3}{2} \quad \tilde{u}_j + \frac{5}{2} \]

Left-biased interpolation

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\(^3\) Deng and Zhang, “Developing high-order weighted compact nonlinear schemes”.
\(^4\) Zhang, Jiang, and Shu, “Development of nonlinear weighted compact schemes with increasingly higher order accuracy”.

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Weighted compact nonlinear scheme (WCNS) \(^3\) \(^4\)

\[ u_j - 2u_j - u_j + 1u_j + 1u_j + 2u_j + 3 \]

\[ \tilde{u}_j - \frac{3}{2} \tilde{u}_j - \frac{1}{2} \tilde{u}_j + \frac{1}{2} \tilde{u}_j + \frac{3}{2} \tilde{u}_j + \frac{5}{2} \]

Right-biased interpolation

\(^3\) Deng and Zhang, “Developing high-order weighted compact nonlinear schemes”.
\(^4\) Zhang, Jiang, and Shu, “Development of nonlinear weighted compact schemes with increasingly higher order accuracy”.

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Weighted compact nonlinear scheme (WCNS)\(^3\)\(^4\)

Riemann solver to get the interface flux

\(^3\)Deng and Zhang, “Developing high-order weighted compact nonlinear schemes”.
\(^4\)Zhang, Jiang, and Shu, “Development of nonlinear weighted compact schemes with increasingly higher order accuracy”.
Overview of the WCNS methodology

Weighted compact nonlinear scheme (WCNS) \(^3\ ^4\)

Finite difference to get the flux derivative

\(^3\) Deng and Zhang, “Developing high-order weighted compact nonlinear schemes”.

\(^4\) Zhang, Jiang, and Shu, “Development of nonlinear weighted compact schemes with increasingly higher order accuracy”.

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The WCHR scheme

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Interpolation

- Critical step since numerical dissipation is controlled by this step
- Typically the limiting step in determining the resolution of the scheme
- Nonlinear WENO style weighting to add dissipation in regions of discontinuities and under-resolved features
Interpolation

• Critical step since numerical dissipation is controlled by this step
• Typically the limiting step in determining the resolution of the scheme
• Nonlinear WENO style weighting to add dissipation in regions of discontinuities and under-resolved features
• Two types of interpolation
  ○ Explicit interpolation
  ○ Compact interpolation
Explicit interpolation (EI)
Explicit interpolation (EI)

\[ EI_0: \tilde{u}_{j+\frac{1}{2}}^{(0)} = \frac{1}{8} (3u_{j-2} - 10u_{j-1} + 15u_j) \]

\[ EI_1: \tilde{u}_{j+\frac{1}{2}}^{(1)} = \frac{1}{8} (-u_{j-1} + 6u_j + 3u_{j+1}) \]

\[ EI_2: \tilde{u}_{j+\frac{1}{2}}^{(2)} = \frac{1}{8} (3u_j + 6u_{j+1} - u_{j+2}) \]

\[ EI_3: \tilde{u}_{j+\frac{1}{2}}^{(3)} = \frac{1}{8} (15u_{j+1} - 10u_{j+2} + 3u_{j+3}) \]

\[ EI_{\text{upwind}} = \sum_{k=0}^{2} d_{k}^{\text{upwind}} EI_k \quad (5\text{th order}); \quad EI_{\text{central}} = \sum_{k=0}^{3} d_{k}^{\text{central}} EI_k \quad (6\text{th order}) \]
Explicit-compact interpolation (ECI)

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Explicit-compact interpolation (ECI)

\[
\begin{align*}
ECI_0 : & \quad \tilde{u}^{(0)}_{j+\frac{1}{2}} = \frac{3}{8} u_{j-2} - \frac{5}{4} u_{j-1} + \frac{15}{8} u_j \\
ECI_1 : & \quad (1 - \xi) \tilde{u}^{(1)}_{j-\frac{1}{2}} + \xi \tilde{u}^{(1)}_{j+\frac{1}{2}} = \left( -\frac{\xi}{2} + \frac{3}{8} \right) u_{j-1} + \frac{3}{4} u_j + \left( \frac{\xi}{2} - \frac{1}{8} \right) u_{j+1} \\
ECI_2 : & \quad \xi \tilde{u}^{(2)}_{j+\frac{1}{2}} + (1 - \xi) \tilde{u}^{(2)}_{j+\frac{3}{2}} = \left( \frac{\xi}{2} - \frac{1}{8} \right) u_j + \frac{3}{4} u_{j+1} + \left( -\frac{\xi}{2} + \frac{3}{8} \right) u_{j+2} \\
ECI_3 : & \quad \tilde{u}^{(3)}_{j+\frac{1}{2}} = \frac{15}{8} u_{j+1} - \frac{5}{4} u_{j+2} + \frac{3}{8} u_{j+3}
\end{align*}
\]

\[ECI_{upwind} = \sum_{k=0}^{2} d_{k}^{\text{upwind}} ECI_k \quad (5^{\text{th}} \text{order}); \quad ECI_{central} = \sum_{k=0}^{3} d_{k}^{\text{central}} ECI_k \quad (6^{\text{th}} \text{order})\]
Displacement relation

Based on linear advection and exact derivatives

- Central linear scheme

\[ \Re(\Phi) \]

\[ \Im(\Phi) \]

- Upwind-biased linear scheme
Nonlinear weighting

- Adaptively add dissipation by using non-linear weighting of the candidate stencils
- We use the LD nonlinear weights

\[ \omega_k = \begin{cases} \sigma \omega_k^{\text{upwind}} + (1 - \sigma) \omega_k^{\text{central}}, & \text{if } R_\tau > \alpha_{RL} \tau, \\ \omega_k^{\text{central}}, & \text{otherwise} \end{cases}, \quad k = 0, 1, 2, 3 \]

---

Flux derivative schemes

- A general compact finite difference scheme:

\[
\alpha f'_{j-1} + f'_j + \alpha f'_{j+1} = a \frac{\tilde{f}_{j+\frac{1}{2}} - \tilde{f}_{j-\frac{1}{2}}}{\Delta x} + b \frac{f_{j+1} - f_{j-1}}{2\Delta x} + c \frac{\tilde{f}_{j+\frac{3}{2}} - \tilde{f}_{j-\frac{3}{2}}}{3\Delta x} + d \frac{\tilde{f}_{j+\frac{5}{2}} - \tilde{f}_{j-\frac{5}{2}}}{5\Delta x}
\]

- Sixth order finite difference forms:
  - Explicit midpoint-to-node differencing (MD):
    \[
    \alpha = 0, \ a = \frac{75}{64}, \ b = 0, \ c = -\frac{25}{128}, \ d = \frac{3}{128}
    \]
  - Explicit midpoint-and-node-to-node differencing (MND):
    \[
    \alpha = 0, \ a = \frac{3}{2}, \ b = -\frac{3}{5}, \ c = \frac{1}{10}, \ d = 0
    \]
  - Compact midpoint-to-node differencing (CMD):
    \[
    \alpha = \frac{9}{62}, \ a = \frac{63}{62}, \ b = 0, \ c = \frac{17}{62}, \ d = 0
    \]
The WCHR scheme

Weighted compact high resolution (WCHR) scheme

- The WCHR6 scheme is a combination of
  - 6\textsuperscript{th} order compact midpoint-to-node finite difference scheme (CMD)
  - 6\textsuperscript{th} order explicit-compact interpolation (ECI)
  - Hybrid central-upwind LD nonlinear weights
Weighted compact high resolution (WCHR) scheme

- The WCHR6 scheme is a combination of
  - $\text{6}^{\text{th}}$ order compact midpoint-to-node finite difference scheme (CMD)
  - $\text{6}^{\text{th}}$ order explicit-compact interpolation (ECI)
  - Hybrid central-upwind LD nonlinear weights

- Other explicit interpolation schemes used for comparison:

<table>
<thead>
<tr>
<th></th>
<th>JS $^6$(EI)</th>
<th>Z $^7$(EI)</th>
<th>LD $^8$(EI)</th>
<th>LD (ECI)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MND ( - - - )</td>
<td>MND-WCNS5-JS</td>
<td>MND-WCNS5-Z</td>
<td>MND-WCNS6-LD</td>
<td>WCHR6</td>
</tr>
<tr>
<td>CMD ( —— )</td>
<td>WCNS5-JS</td>
<td>WCNS5-Z</td>
<td>WCNS6-LD</td>
<td></td>
</tr>
</tbody>
</table>

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8 Wong and Lele, “High-order localized dissipation weighted compact nonlinear scheme for shock-and interface-capturing in compressible flows”.

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The WCHR scheme

Derivative scheme

Weighted compact high resolution (WCHR) scheme

- The WCHR6 scheme is a combination of
  - $6^{th}$ order compact midpoint-to-node finite difference scheme (CMD)
  - $6^{th}$ order explicit-compact interpolation (ECI)
  - Hybrid central-upwind LD nonlinear weights

- Other explicit interpolation schemes used for comparison:

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<td>WCHR6</td>
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</table>

- For all the test problems, only the schemes with the compact finite difference scheme (CMD) are used to highlight the importance of the compact interpolation

- $\xi = 2/3$ is used for all the cases with ECI

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8 Jiang and Shu, “Efficient implementation of weighted ENO schemes”.
8 Borges et al., “An improved weighted essentially non-oscillatory scheme for hyperbolic conservation laws”.
8 Wong and Lele, “High-order localized dissipation weighted compact nonlinear scheme for shock-and interface-capturing in compressible flows”.

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Approximate dispersion relation

- The dispersion relation plots earlier are for the linear schemes.
- Characterizing the dispersion relation is harder for the full non-linear scheme.
The dispersion relation plots earlier are for the linear schemes.

Characterizing the dispersion relation is harder for the full non-linear scheme.

Can approximate the dispersion relation using the method proposed by Pirozzoli\(^9\).

Main idea is to numerically compute the dispersion relation for each wavenumber using a monochromatic wave with that wavelength.

---

Approximate dispersion relation

- WCHR has highest resolution
- Since the linear version of scheme has higher resolution, more localized dissipation can be used
Dispersion error

- WCHR6 has highest resolving efficiency with a threshold of $10^{-2}$
Extension to Euler equations

- In extending the scheme to the Euler equations, there are three choices
  1. Interpolate conservative variables
  2. Interpolate primitive variables
  3. Interpolate characteristic variables
- Interpolating characteristic variables is preferable in order to prevent the interaction of discontinuities in different characteristic fields
- Leads to more precise addition of numerical dissipation

---

The WCHR scheme

Extension to Euler equations

- But characteristic interpolation requires solving a larger system
- Ensures consistency in transforming to and back from the characteristic space

\[
\alpha_{j+\frac{1}{2}} A_{j+\frac{1}{2}}^RL \cdot \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_{j-\frac{1}{2}} + \beta_{j+\frac{1}{2}} A_{j+\frac{1}{2}}^RL \cdot \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_{j+\frac{3}{2}} + \gamma_{j+\frac{1}{2}} A_{j+\frac{1}{2}}^RL \cdot \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_{j+\frac{3}{2}} =
\]

\[
\alpha_{j+\frac{1}{2}} A_{j+\frac{1}{2}}^RL \cdot \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_{j-1} + \beta_{j+\frac{1}{2}} A_{j+\frac{1}{2}}^RL \cdot \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_{j} + \gamma_{j+\frac{1}{2}} A_{j+\frac{1}{2}}^RL \cdot \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_{j+1} + \delta_{j+\frac{1}{2}} A_{j+\frac{1}{2}}^RL \cdot \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_{j+2}
\]
Extension to Euler equations

- But characteristic interpolation requires solving a larger system
- Ensures consistency in transforming to and back from the characteristic space

\[
\begin{align*}
\alpha_j \frac{1}{2} A_{RL}^{j+1/2} \cdot \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_{j-1/2} + \beta_j \frac{1}{2} A_{RL}^{j+1/2} \cdot \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_{j+1/2} + \gamma_j \frac{1}{2} A_{RL}^{j+1/2} \cdot \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_{j+3/2} = \\
\alpha_j \frac{1}{2} A_{RL}^{j+1/2} \cdot \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_{j-1} + b_j \frac{1}{2} A_{RL}^{j+1/2} \cdot \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_j + c_j \frac{1}{2} A_{RL}^{j+1/2} \cdot \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_{j+1} + d_j \frac{1}{2} A_{RL}^{j+1/2} \cdot \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_{j+2}
\end{align*}
\]

- Block-tridiagonal matrix in primitive variables instead of 3 independent tridiagonal systems in characteristic space
Characteristics based interpolation

- Need to do characteristic decomposition implicitly
- Block tri-diagonal system instead of 3 tri-diagonal systems
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Advection of a sinusoidal entropy wave

- Advection of a sinusoidal entropy wave in a periodic domain $x \in [0.0, 1.0)$ with initial conditions

$$ (\rho, u, p) = (1 + 0.5 \sin (\pi x), 1, 1) $$

- WCHR6 is 6th order accurate.
- WCHR6 has $\approx 5$ times smaller error than WCNS6-LD.
Advection of a broadband entropy disturbance

- Density of uniform flow disturbed by a broadband signal

\[
\begin{pmatrix}
\rho \\
u \\
p
\end{pmatrix} = \begin{pmatrix}
1 + \delta \sum_{k=1}^{N/2} (E_{\rho}(k))^{1/2} \sin(2\pi k (x + \psi_k)) \\
1 \\
1
\end{pmatrix}
\]

- Broadband of density disturbances with power spectral density

\[
E(k) = \left(\frac{k}{k_0}\right)^4 \exp \left\{-2 \left(\frac{k}{k_0}\right)^2\right\}
\]

with \( k_0 = 10 \)

- The problem is solved using 128 points on a periodic domain \( x \in [0.0, 1.0) \)
Advection of a broadband entropy disturbance

- The density field after one period:
Results
1D test problems

Advection of a broadband entropy disturbance

- WCNS5-JS and WCNS5-Z have high dissipation errors
- WCNS6-LD has lower dissipation, but still distorts the spectrum
- WCHR6 has the least dissipation and spectral distortion
The Sod shock-tube problem has an initial pressure and density discontinuity given by

\[
(\rho, u, p) = \begin{cases} 
(1, 0, 1), & x < 0 \\
(0.125, 0, 0.1), & x \geq 0
\end{cases}
\]

The problem is solved using 100 points on a domain \( x \in [-0.5, 0.5] \).
All scheme can capture the shock well. WCNS6-LD and WCHR6 have sharpest gradients.
Shu-Osher problem

- This problem involves the interaction of a Mach 3 shock wave with an entropy wave. Initial conditions are

\[
(\rho, u, p) = \begin{cases}
(27/7, 4\sqrt{35}/9, 31/3), & x < -4 \\
(1 + 0.2 \sin (5x), 0, 1), & x \geq -4
\end{cases}
\]

- The problem is solved using 150 points on a domain \( x \in [-5, 5] \)
Shu-Osher problem

- With 150 points

- WCNS5-JS and WCNS5-Z (upwind-biased schemes) have much larger dissipation errors
- WCNS6-LD has larger dispersion error than WCHR6
Shock-vortex interaction

- A Mach 1.2 shock interaction with a strong vortex of vortex Mach number $M_v = 1$
- Domain size is chosen to be $[-D, D) \times [-D/2, D/2)$, where $D = 40$
- Periodic boundary conditions used in both $x$ and $y$ directions
- A 2D grid with $1024 \times 512$
Shock-vortex interaction

\[ \frac{(p - p_\infty)}{(\rho_\infty c_\infty^2)} \]
Results

2D test problems

Pressure at $t = 6$

WCNS5-JS

WCNS5-Z

WCNS6-LD

WCHR6

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Results

2D test problems

Sound pressure at $t = 16$

WCNS5-JS

WCNS5-Z

WCNS6-LD

WCHR6

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Sound pressure at radial direction

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Taylor-Green Vortex

- Inviscid vortex breakdown in a periodic domain with size $[0, 2\pi)^3$
- The initial conditions are given by:

$$\begin{pmatrix}
\rho \\
u \\
v \\
w \\
p
\end{pmatrix} = 
\begin{pmatrix}
1 \\
sin x \cos y \cos z \\
- \cos x \sin y \cos z \\
0 \\
100 + \frac{(\cos (2z) + 2)(\cos (2x) + \cos (2y)) - 2}{16}
\end{pmatrix}$$

- Flow is essentially incompressible
- Grid resolutions of $32^3$ and $64^3$ are considered
- No sub-grid model is used in order to assess the dissipation characteristics of the scheme itself
Taylor-Green Vortex

- $32^3$ problem

- WCHR is the least dissipative based on the TKE plot
- WCHR also has the most enstrophy and matches the semi-analytical result\(^{11}\) for longer time

Taylor-Green vortex

- $64^3$ problem

- On a $64^3$ grid, WCHR captures much more of the enstrophy and is comparable to the LAD methods presented in Johnsen et. al.\textsuperscript{12}

• $64^3$, spectra at $t = 8$

Velocity energy spectrum

Vorticity energy spectrum
Compressible homogeneous isotropic turbulence

- Decaying compressible homogeneous isotropic turbulence

\[ E(k) \propto k^4 \exp \left( -2 \left( \frac{k}{k_0} \right)^2 \right), \quad k_0 = 4 \]

- Initial turbulent Mach number \( M_t = 0.6 \)
- Initial Taylor Reynolds number \( \text{Re}_\lambda = 100 \)
- High turbulent Mach number creates eddy shocklets in the domain

- Good test case to assess the numerical dissipation characteristics of different schemes
- No sub-grid model is used in order to assess the dissipation characteristics of the scheme itself
- Solved on a \( 64^3 \) grid

---

Compressible homogeneous isotropic turbulence

Numerical schlieren on a slice

1σ enstrophy contour (blue) and $-3\sigma$ dilatation (red) contours at $t/\tau = 2$
• Spectrally filtered DNS solution from Johnsen et. al.\textsuperscript{14} in open circles

\textsuperscript{14} Johnsen et al., “Assessment of high-resolution methods for numerical simulations of compressible turbulence with shock waves”.

\cite{johnsen2014assessment}
Compressible homogeneous isotropic turbulence

- Spectra at $t/\tau = 2$

Velocity energy spectrum

Vorticity energy spectrum
Compressible homogeneous isotropic turbulence

- Spectra at \( t/\tau = 2 \)

Density energy spectrum

Dilatation energy spectrum
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Regent task based system

- Regent is a language for implicit dataflow parallelism
- Discovers dataflow parallelism in sequential code by computing a dependence graph over tasks
- Programmer only expresses the parallelism but does not explicitly implement any communication, etc.
- Data transfer between tasks is handled by the runtime system
- Works for node-node and CPU-GPU data transfers
For a 3D problem, the cost per grid point for the interpolation in one direction is given below:

<table>
<thead>
<tr>
<th>Operation Counts</th>
<th>EI char.</th>
<th>ECI prim.</th>
<th>ECI char.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matrix solve</td>
<td>11</td>
<td>45</td>
<td>195</td>
</tr>
<tr>
<td>Interpolation RHS</td>
<td>55</td>
<td>55</td>
<td>55</td>
</tr>
<tr>
<td>Characteristic decom-</td>
<td>66</td>
<td>0</td>
<td>66</td>
</tr>
<tr>
<td>position</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Smoothness indicators</td>
<td>440</td>
<td>440</td>
<td>440</td>
</tr>
<tr>
<td>Nonlinear weights</td>
<td>630</td>
<td>720</td>
<td>720</td>
</tr>
<tr>
<td>Total</td>
<td>1202</td>
<td>1260</td>
<td>1476</td>
</tr>
</tbody>
</table>

ECI with characteristic decomposition is about 23% more expensive in terms of operation count.
Block-tridiagonal solve

• The main bottleneck in performance is the block tridiagonal matrix system

• Use the SuperLU\textsuperscript{15} library to solve the block-tridiagonal matrix system

• Do symbolic factorization once and use sparsity pattern multiple times

• With SuperLU, the block tridiagonal solves take $\approx 80 - 90\%$ of the total wall clock time

• Not consistent with Flop count estimates from before

• Need to perform some data copies to be able to interface with SuperLU, but that still does not explain the high cost
Custom block-tridiagonal solver

• Consider a block tridiagonal matrix system $Ax = b$

$$
\begin{pmatrix}
\tilde{\beta}_1 & \tilde{\gamma}_1 \\
\tilde{\alpha}_2 & \tilde{\beta}_2 & \tilde{\gamma}_2 \\
\vdots & & \ddots \\
\tilde{\alpha}_{N-1} & \tilde{\beta}_{N-1} & \tilde{\gamma}_{N-1} \\
\tilde{\alpha}_N & \tilde{\beta}_N
\end{pmatrix}
\begin{pmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\vdots \\
\tilde{x}_{N-1} \\
\tilde{x}_N
\end{pmatrix}
= 
\begin{pmatrix}
\tilde{b}_1 \\
\tilde{b}_2 \\
\vdots \\
\tilde{b}_{N-1} \\
\tilde{b}_N
\end{pmatrix}
$$

• $\tilde{\alpha}_i$, $\tilde{\beta}_i$ and $\tilde{\gamma}_i$ are $5 \times 5$ matrices corresponding to the Jacobian of conserved variables w.r.t primitive variables

• $x_i$ and $b_i$ are $5 \times 1$ vector elements of the solution vector and the RHS vector respectively
Custom block-tridiagonal solver

- Speed up the block-tridiagonal solver performance using a custom solver
- The Jacobian matrix forming each block in the $x$ interpolation is

$$
\begin{pmatrix}
0 & -\frac{\rho c}{2} & 0 & 0 & \frac{1}{2} \\
1 & 0 & 0 & 0 & -\frac{1}{c^2} \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & \frac{\rho c}{2} & 0 & 0 & \frac{1}{2}
\end{pmatrix}
$$

- Idea is to decouple $v$ and $w$ interpolations to get a $3 \times 3$ block tridiagonal system and 2 tridiagonal systems instead of a $5 \times 5$ block tridiagonal system
- Cost of each block inversion reduces from $O(5^3)$ to $O(3^3)$
Custom block-tridiagonal solver

- Use approach similar to the regular Thomas algorithm to derive a block-tridiagonal algorithm
- Forward elimination

\[ \Delta_i = \left[ \tilde{\beta}_i - \tilde{\alpha}_i \Delta_{i-1} \tilde{\gamma}_{i-1} \right]^{-1} \]

\[ \hat{b}_i = b_i - \tilde{\alpha}_i \Delta_{i-1} \hat{b}_{i-1} \]

- Back substitution

\[ x_i = -\Delta_i \left[ -\hat{b}_i + \tilde{\gamma}_i x_{i+1} \right] \]

- Use Sherman-Morrison low rank correction for periodic problems

\[ \left( A + UV^T \right)^{-1} = A^{-1} - A^{-1} U \left( I + V^T A^{-1} U \right)^{-1} V^T A^{-1} \]
The custom algorithm was implemented in Regent without any specific optimizations

\ (~ 30\times \) speedup in the matrix solves

\ (~ 8\times \) speedup of the full interpolation algorithm

For full 3D taylor-green vortex on a $64^3$ grid

- Old implementation took $\sim 73s$ per time step
- New implementation takes $\sim 15s$ per time step
- HAMeRS\textsuperscript{16} (fully explicit scheme) takes $\sim 7.5s$ per time step

Being within a factor of 2 compared to HAMeRS is quite good since no vectorization optimizations have been performed

\textsuperscript{16}https://fpal.stanford.edu/hamers
Outline

1. Introduction
2. Overview of the WCNS methodology
   - Explicit interpolation
   - Explicit-compact interpolation
   - Nonlinear weights
   - Derivative scheme
   - Approximate dispersion relation
   - Extension to Euler equations
3. The WCHR scheme
4. Results
   - 1D test problems
   - 2D test problems
   - 3D test problems
5. Implementation details
6. Conclusions
Conclusions

- WCHR6 has higher resolution than other WCNS’s due to better spectral properties of the nonlinear explicit-compact interpolation.
- Computational cost in terms of operation count was shown to be only $\sim 23\%$ more than traditional WCNS’s.
- 1D and 2D problems show the robustness of WCHR6 in capturing shocks and high wavenumber features.
- 3D problems highlight the minimal dissipation characteristic of the WCHR6 compared to other WCNS’s.
- Future work:
  - Develop consistent and stable boundary schemes.
  - Analyze space-time characteristics of WCHR6.
  - Parallel algorithms for the block tri-diagonal solve.
Questions?