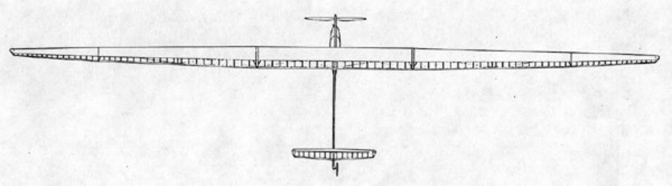
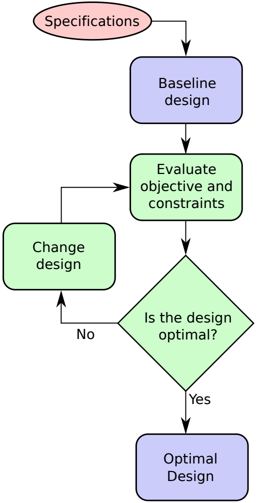


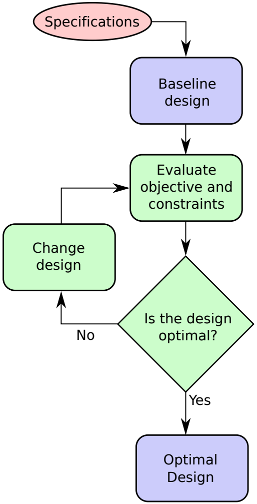
Exploiting convex structure in aircraft design optimization

Warren Hoburg
Massachusetts Institute of Technology
with Philippe Kirschen, Cody Karcher, and Ned Burnell

NASA Ames Applied Modeling and Simulation Seminar
June 5, 2015



[Adapted from Alonso 2012]



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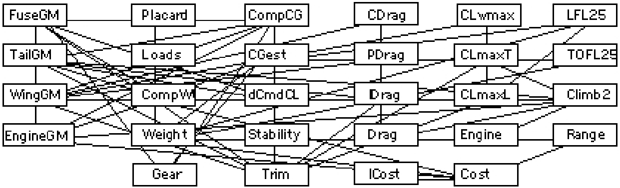
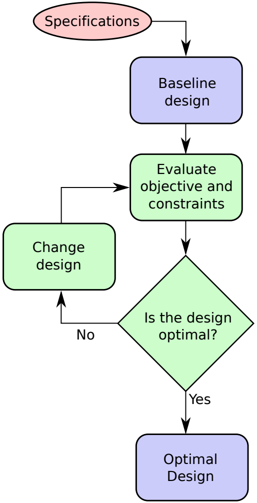
Challenge #1: Black box models

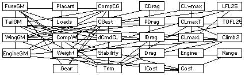


[Kroo 1994]

- Disciplinary separations
- Expensive function evaluations

Challenge #1: Black box models





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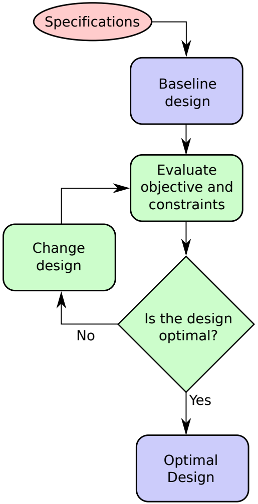

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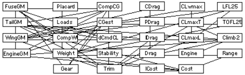
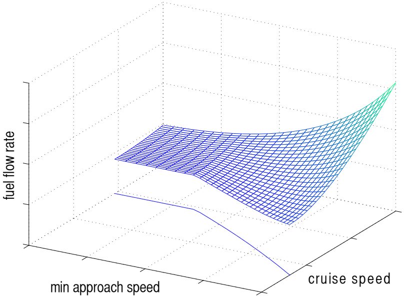


Challenge #2: Design tradeoffs



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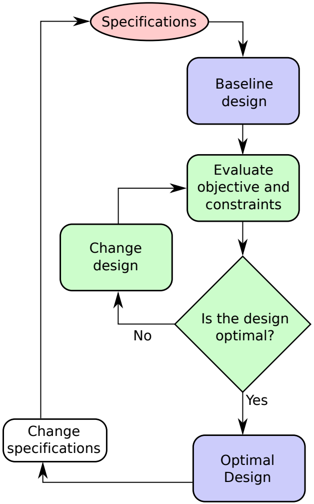

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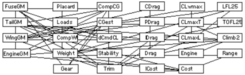

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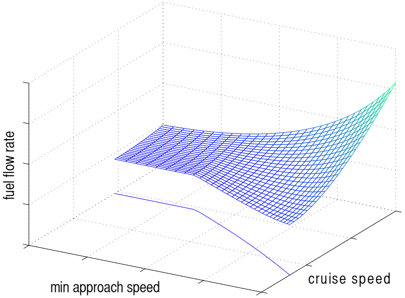

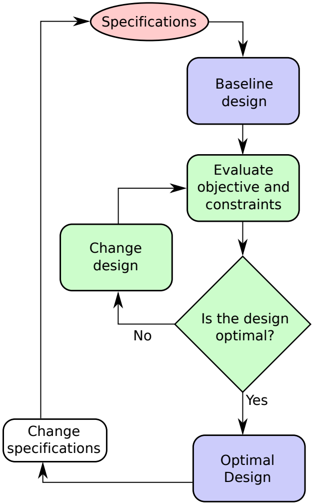

[Adapted from Alonso 2012]

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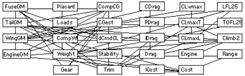
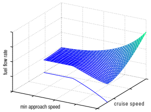
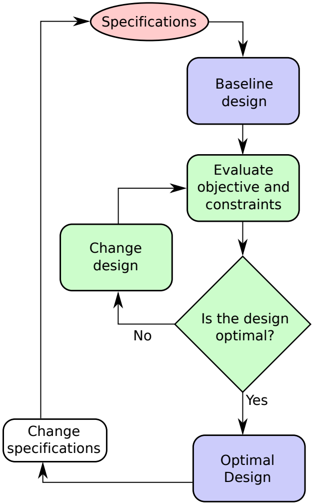


Challenge #2: Design tradeoffs

- Want Pareto frontier
  $\rightarrow$ must solve many times





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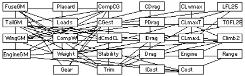

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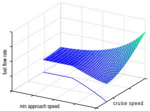
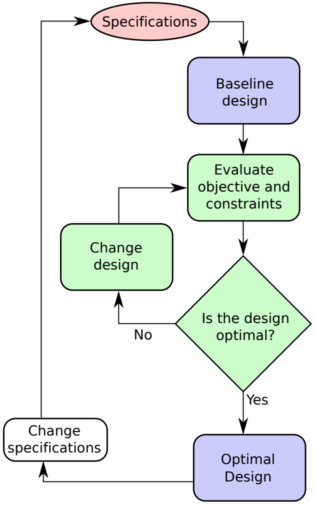
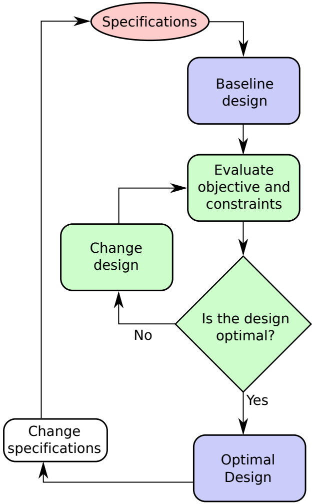


Challenge #3: Optimization issues

- Local vs. global optima
- Sensitivity to initial guess
- Sensitivity to solver params



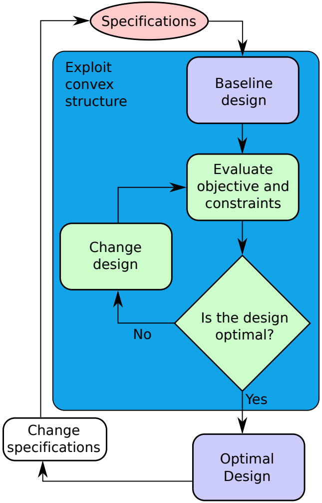
[Adapted from Alonso 2012]



[Adapted from Alonso 2012]

Insight

- Surprisingly, many relationships in engineering design have an underlying convex structure.



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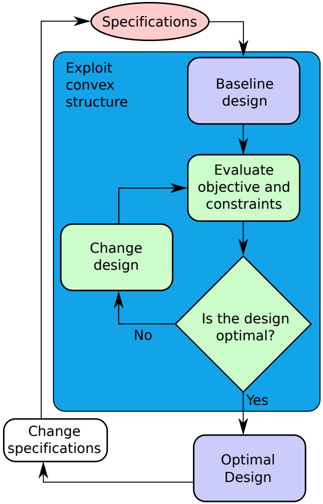


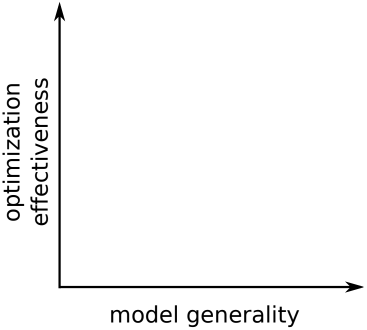
[Adapted from Alonso 2012]

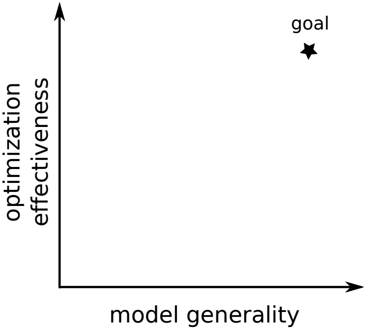
Insight

- Surprisingly, many relationships in engineering design have an underlying convex structure.

Benefits

- Globally optimal solutions
- Numerically stable algorithms
- No initial guesses; no solver parameter tuning
- Extremely fast solutions, even for large problems


optimization effectiveness
model generality


goal
optimization effectiveness
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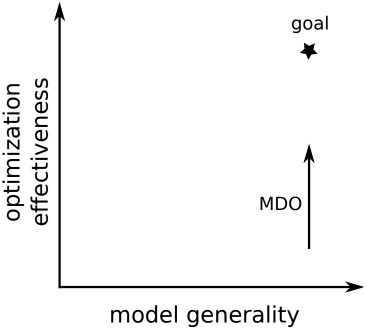
goal
MDO
optimization effectiveness
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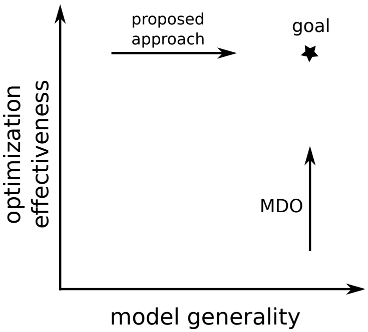

proposed
approach
goal
MDO
optimization
effectiveness
model generality

Today's talk

Geometric programming overview

Aircraft design modeling examples

Current and future directions

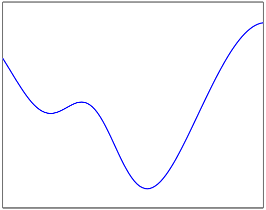
Today's talk

Geometric programming overview

Aircraft design modeling examples

Current and future directions

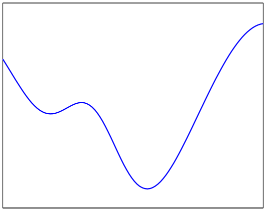
General nonlinear program



$$
\begin{aligned}
\text{minimize} \quad & f_0(\boldsymbol{x}) \\
\text{subject to} \quad & f_i(\boldsymbol{x}) \leq 0, \quad i = 1, ..., N_i \\
& h_j(\boldsymbol{x}) = 0, \;\; j = 1, ..., N_e
\end{aligned}
$$

- In general, extremely difficult to solve

General nonlinear program



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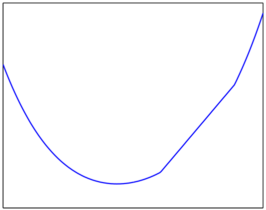
Convex program



Same as nonlinear program, except

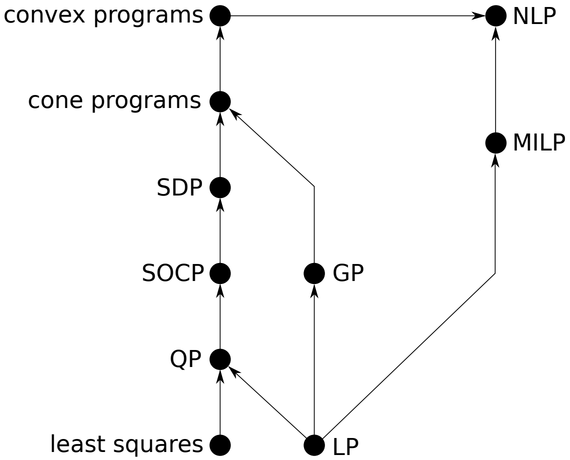
- $f_i(\boldsymbol{x})$ must be convex
- $h_j(\boldsymbol{x})$ must be affine

- Very efficient to solve

"... the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity."

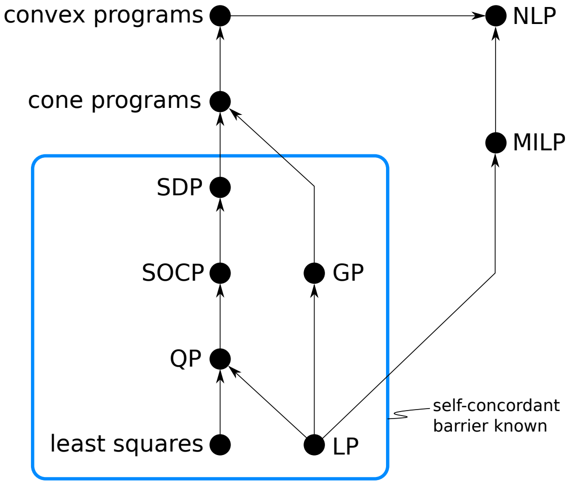
— R. Rockafeller, SIAM Review 1993

Optimization hierarchy



Adapted from Boyd 2002

Optimization hierarchy



Adapted from Boyd 2002

Geometric program: definition

Geometric program: definition

Monomial function

$$m(\boldsymbol{x}) = c\prod_{i=1}^{n} x_i^{a_i}, \qquad c > 0 \qquad\qquad \text{(e.g., } \tfrac{1}{2}\rho V^2 C_L S\text{)}$$

Geometric program: definition

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Posynomial function: sum of monomials

$$p(\boldsymbol{x}) = \sum_{k=1}^{K} c_k \prod_{i=1}^{n} x_i^{a_{ik}}, \;\; c_k > 0 \quad \text{(e.g., } c_d + \frac{C_L^2}{\pi e A}\text{)}$$

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Geometric program (GP)

$$\boxed{\begin{array}{lll} \text{minimize} & p_0(\boldsymbol{x}) & \\ \text{subject to} & p_i(\boldsymbol{x}) \le 1, & i = 1, \ldots, N_i, \\ & m_j(\boldsymbol{x}) = 1, & j = 1, \ldots, N_e \end{array}}$$

with p_i posynomial, m_i monomial

$\boldsymbol{x} = (x_1, x_2, \ldots, x_n) > 0$

Geometric program: convex formulation

variable change: $y_i := \log x_i$

Geometric program: convex formulation

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- Monomials $m(\boldsymbol{x}) = c\prod_{i=1}^n x_i^{a_i}$: affine in $\boldsymbol{y}$ after log transform

$$\log m = b + a^T\boldsymbol{y} \qquad (b = \log c)$$

Geometric program: convex formulation

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$$\log p = \log \left(\sum_{k=1}^{K} e^{b_k + a_k^T \boldsymbol{y}} \right)$$

Geometric program: convex formulation

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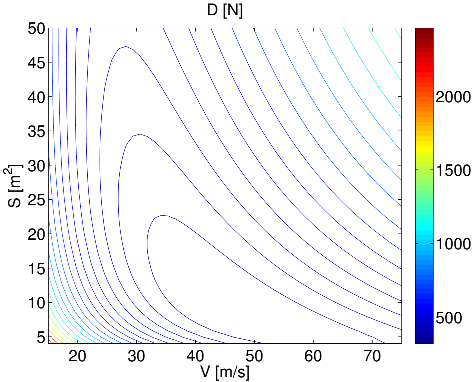
$$\log p = \log \left(\sum_{k=1}^K e^{b_k + a_k^T \boldsymbol{y}} \right)$$

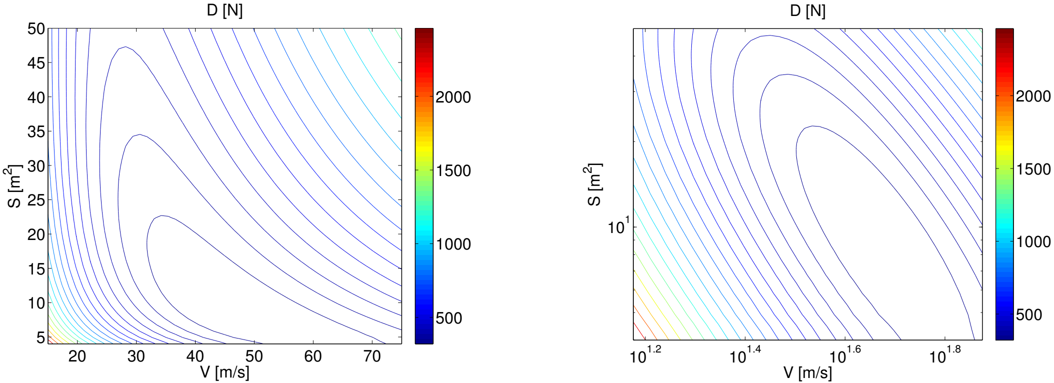
- GP in convex form

$$\boxed{\begin{array}{ll} \text{minimize} & \log \left(\sum_{k=1}^{K_0} \exp(b_{0k} + a_{0k}^T \boldsymbol{y}) \right) \\ \text{subject to} & \log \left(\sum_{k=1}^{K_i} \exp(b_{ik} + a_{ik}^T \boldsymbol{y}) \right) \le 0, \quad i = 1, \ldots, N_p \\ & G\boldsymbol{y} + h = 0 \end{array}}$$

Geometric program: convex formulation

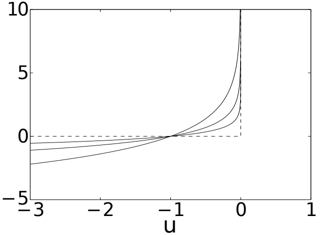


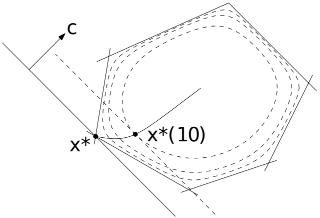
Geometric program: convex formulation



Solving geometric programs

Interior-point methods





Figures: [Boyd 2004]

Benefits:

- Globally optimal solution, guaranteed
- Robust: no initial guesses or parameter tuning
- Off-the-shelf solvers

Boyd GP benchmarks (2005) [1]

- dense GP: 1,000 variables, 10,000 constraints: less than 1 minute
- sparse GP: 10,000 variables, 1,000,000 constraints: "minutes"

Today's talk

Geometric programming overview

Aircraft design modeling examples

Current and future directions

Modeling warm-up: monomial examples

- Steady level flight relations

$$W = \frac{1}{2}\rho V^2 C_L S, \qquad T = \frac{1}{2}\rho V^2 C_D S, \qquad TV = h_{\mathsf{fuel}} \dot{m}_{\mathsf{fuel}} \eta_{\mathsf{thm}} \eta_{\mathsf{eng}} \eta_{\mathsf{prop}}$$

- Non-dimensional coefficients

$$\mathsf{Re} = \frac{\rho V c}{\mu}, \quad M = \frac{V}{\sqrt{\gamma R T}}$$

- Sizing parameters

$$\tau = \frac{t}{c}, \quad \lambda = \frac{c_t}{c_r}, \quad A = \frac{b^2}{S}$$

- Empirical power law models

$$W_{\substack{\mathsf{main}\\ \mathsf{gear}}} = 0.011 W^{0.888} N_{\mathsf{land}}^{0.25} L_{\mathsf{main}}^{0.4} N_{\mathsf{wheel}}^{0.321} N_{\mathsf{ss}}^{-0.5} V_{\mathsf{stall}}^{0.1}$$ [Raymer 2006]

Modeling example 1: Taylor expansion

Breguet range equation

$$R = \frac{h_f}{g}\eta_0\frac{C_L}{C_D}\log(1 + \frac{W_{fuel}}{W_{empty}})$$

Modeling example 1: Taylor expansion

Breguet range equation

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$$\exp(\frac{RgC_D}{h_f C_L \eta_0}) = 1 + \frac{W_{fuel}}{W_{empty}}$$

Modeling example 1: Taylor expansion

Breguet range equation

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$$\exp(\frac{RgC_D}{h_f C_L \eta_0}) = 1 + \frac{W_{fuel}}{W_{empty}}$$

GP formulation

$$z \geq \frac{RgC_D}{h_f C_L \eta_0}$$

$$z + \frac{z^2}{2!} + \frac{z^3}{3!} + ... \leq \frac{W_{fuel}}{W_{empty}}$$

Modeling example 1: Taylor expansion

Breguet range equation

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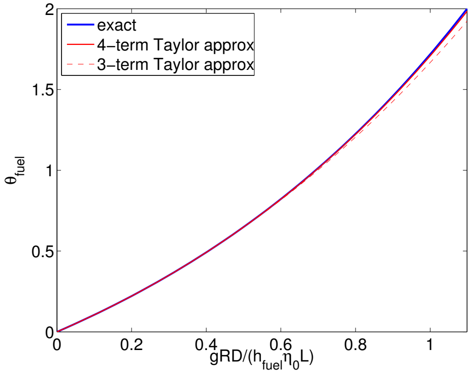
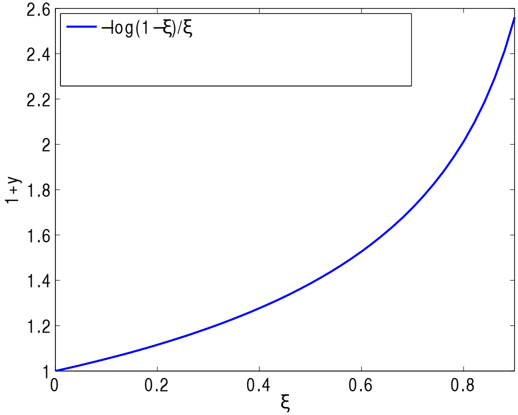
$$z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \leq \frac{W_{fuel}}{W_{empty}}$$



Modeling example 2: Implicit posynomial fitting

$$y = -\frac{\log(1-\xi)}{\xi}$$



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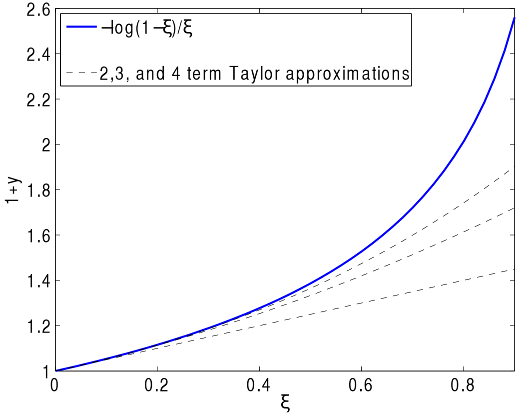
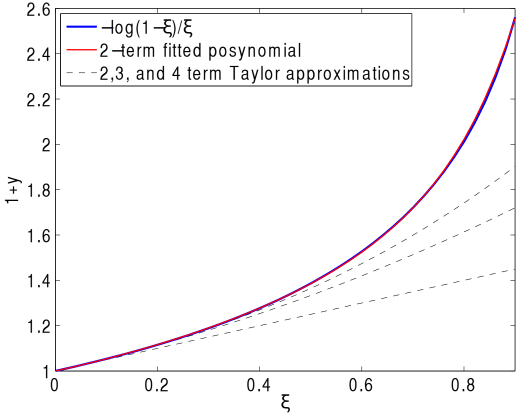


Modeling example 2: Implicit posynomial fitting

$$y = -\frac{\log(1-\xi)}{\xi}$$

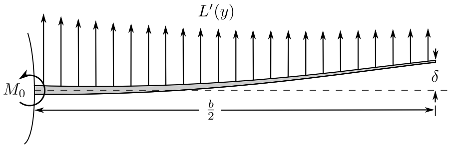
$$1 \geq \frac{0.0464\xi^{2.73}}{y^{2.88}} + \frac{1.044\xi^{0.296}}{y^{0.049}}$$



Modeling example 3: Local function approximation

Stress limit:

$$\sigma_{safe} S_{root} \geq N_{lift} M_{root}$$



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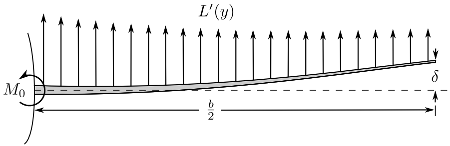

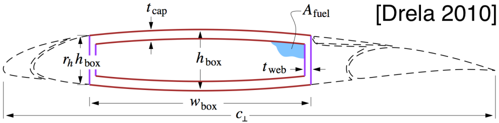

[Drela 2010]

$$\bar{h}_{rms} \bar{w} \bar{t}_{cap}^2 + \bar{I}_{cap} \leq \frac{1}{2} \bar{w} \bar{t}_{cap}$$

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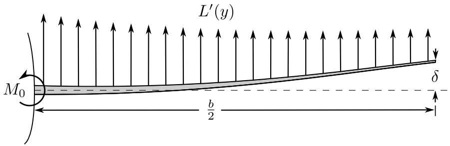
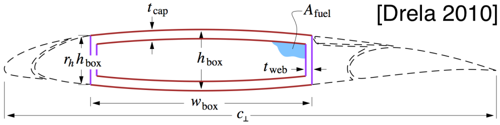


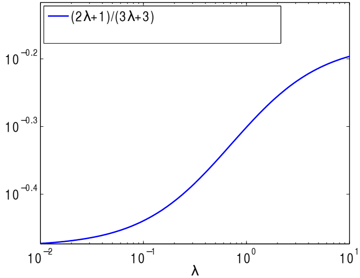


[Drela 2010]

$$\bar{h}_{rms} \bar{w} \bar{t}_{cap}^2 + \bar{I}_{cap} \leq \frac{1}{2} \bar{w} \bar{t}_{cap}$$

Applied root moment:

$$M_0 \geq \frac{\tilde{W} b}{4} \left[\frac{2\lambda + 1}{3\lambda + 3} \right]$$



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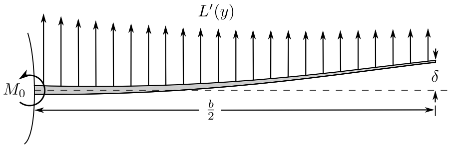
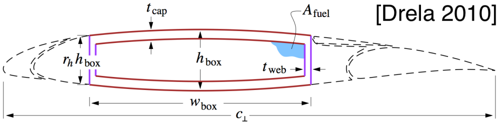
Stress limit:

$$\sigma_{safe} S_{root} \geq N_{lift} M_{root}$$



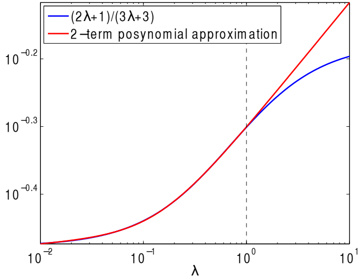


[Drela 2010]

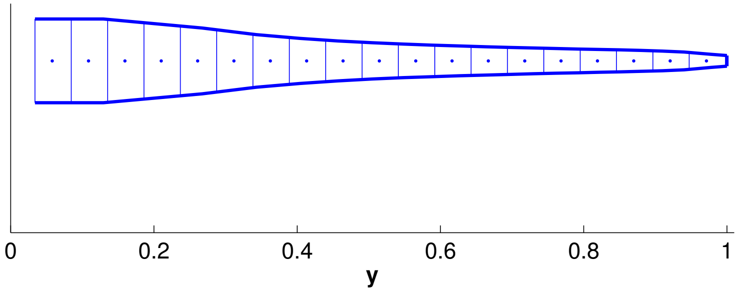
$$\bar{h}_{rms} \bar{w} \bar{t}_{cap}^2 + \bar{I}_{cap} \leq \frac{1}{2} \bar{w} \bar{t}_{cap}$$

Applied root moment:

$$M_0 \geq \frac{\tilde{W} b}{4} \left[\frac{2\lambda + 1}{3\lambda + 3} \right]$$



Modeling example 4: spanwise discretization



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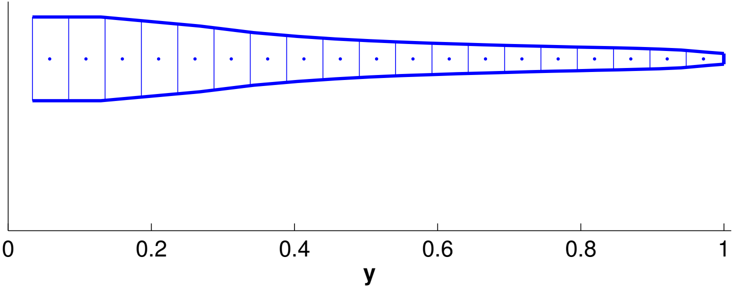

Problem: $C_P \leq (\Delta C_P)_1 + (\Delta C_P)_2 + \ldots + (\Delta C_P)_N \quad ?$

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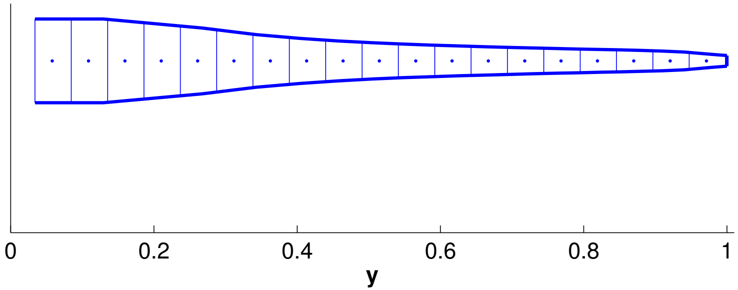

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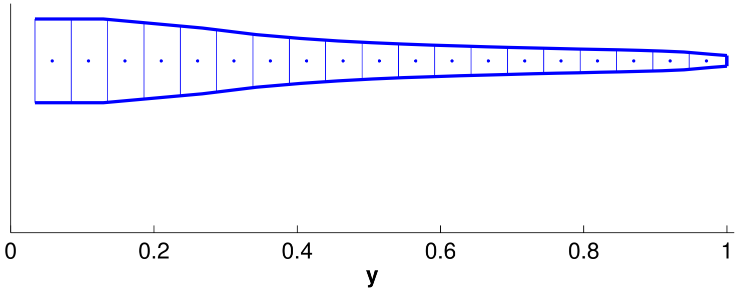

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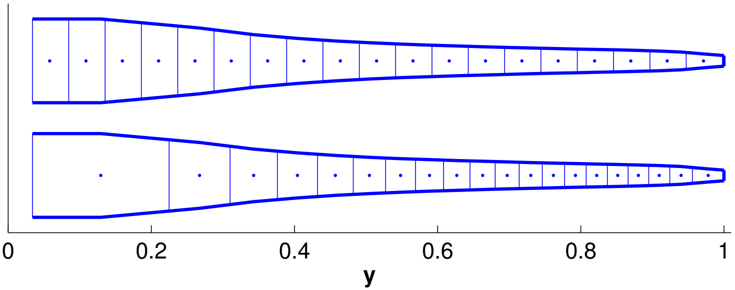

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Modeling example 5: Fitting models from data

$\sim 10,000$ data points from

$$c_d(C_L, \mathrm{Re}, \tau)$$

for NACA-24xx airfoils, generated using XFOIL [Drela 1989]

- τ ranging from 8% to 16%
- Re ranging from 10^6 to 10^7 (small homebuilt to small jet)
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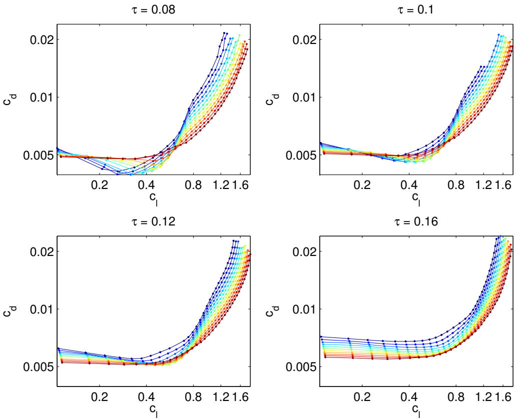

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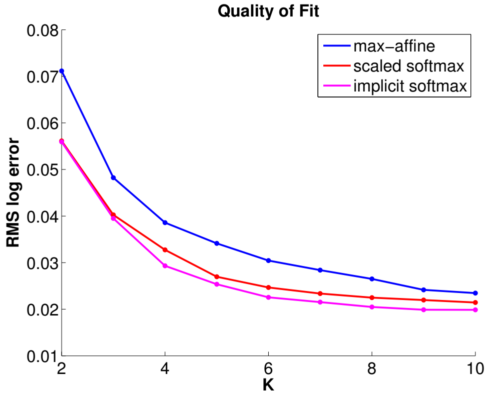


GPkit: GP modeling in python

- substitution instead of constants

```
gpkit.Variable("R", 8, "meters")
```

- unit checking and conversions

```
gpkit.Variable("W", 4.94, "kilonewtons")
```

- interactive explorations

```
gpkit.interactive.widget(gp)
```

- sweeps over the design space

```
gp.sub("R", ("sweep", [4, 6, 8]))
```

- Experimental: optimization involving random variables

```
gpkit.Variable("\sigma_max", min=220, expected=276, "MPa")
```

http://gpkit.readthedocs.org

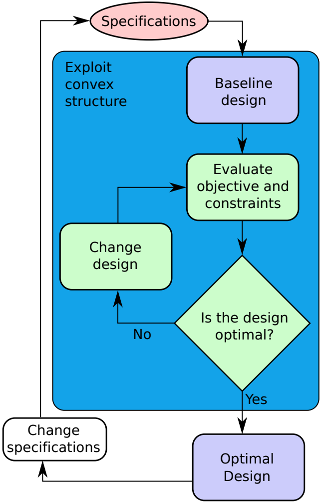
Today's talk

Geometric programming overview

Aircraft design modeling examples

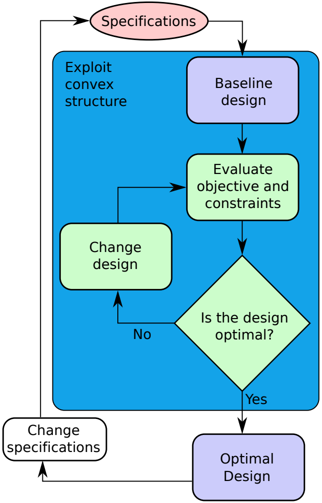
Current and future directions

Constraint sensitivities



Which trade studies should we conduct?

Constraint sensitivities



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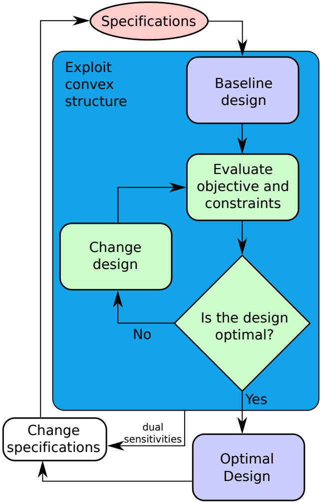

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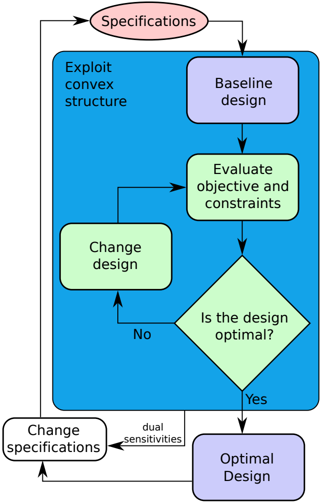

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Example The dual variable associated with constraint i is -0.27.

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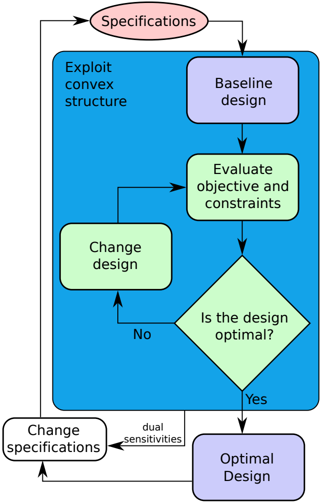

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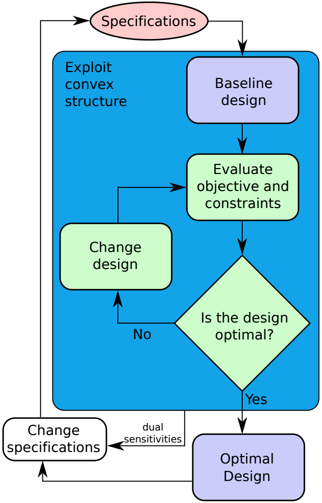

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Applications

- Guide trade studies
- Direct engineering effort to most important areas
- Better understand uncertainty propagation

Feasibility Analysis

When constraints cannot all be satisfied, GP solvers provide a mathematical certificate that no feasible point exists.

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Original GP

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The closest feasible point GP is always* feasible, and its optimal point is within $100(s-1)$% of satisfying the original inequality constraints.

*assuming monomial equality constraints are feasible

Signomial programming

- Primary limitation of GP approach: models must be log-convex
- Can handle more general models using signomial programming

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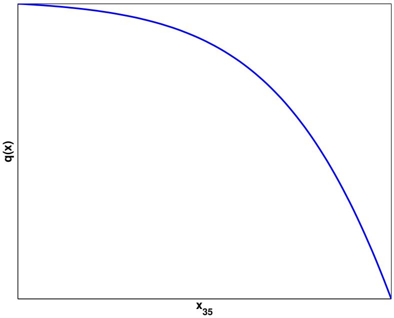
Sketch of sequential GP approach



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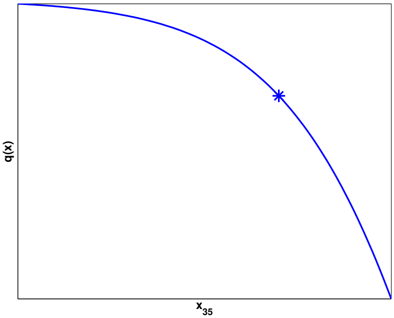

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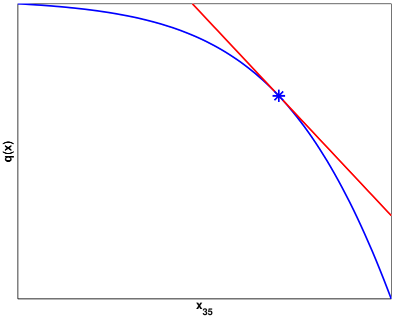

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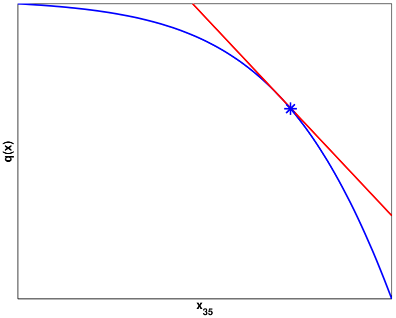

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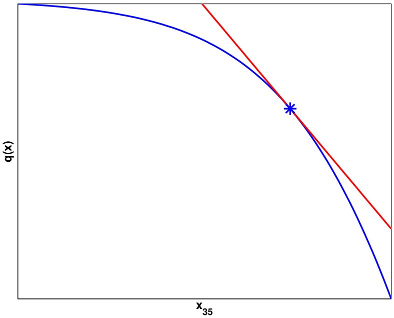

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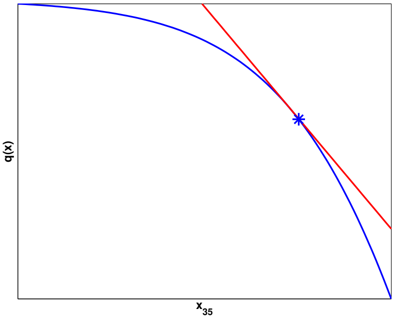

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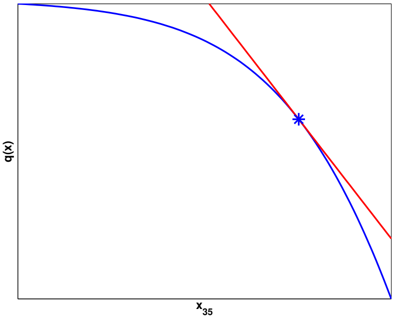

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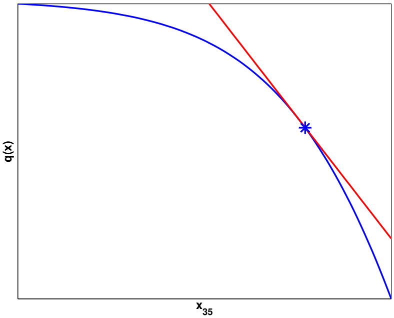

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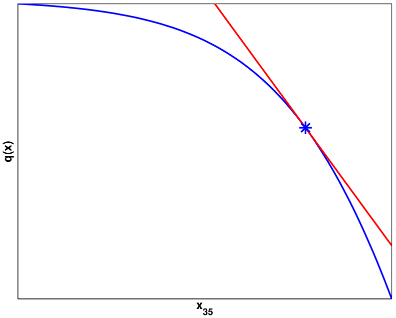

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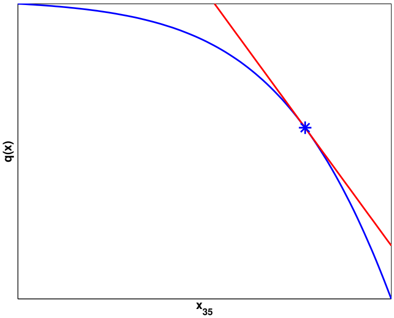
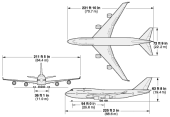


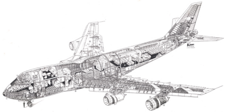
System-level design

Conceptual Design



Preliminary Design

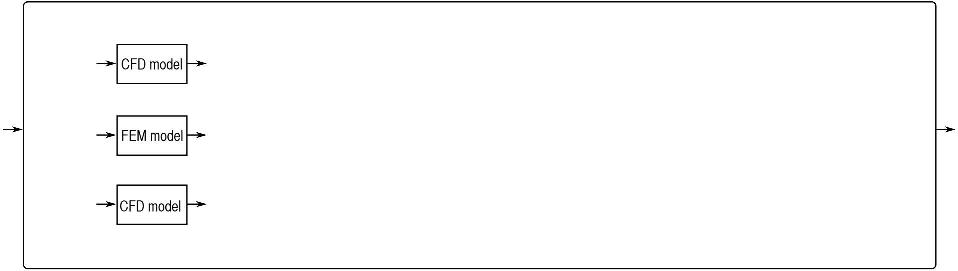
Detailed Design



Manufacturing

Operations

System-level design



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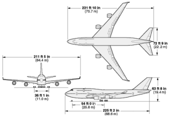

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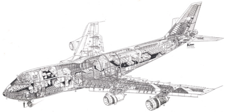

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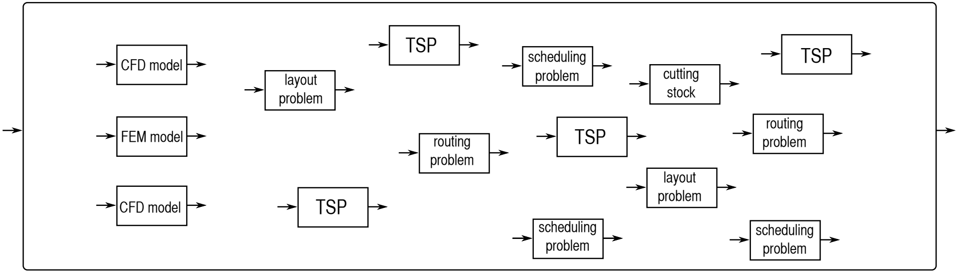

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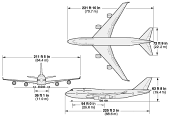

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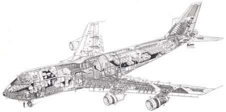

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Take-aways

- Importance of mathematical structure
- Key to tractability: convexity
- Result: reliable and efficient optimization that scales to large problems

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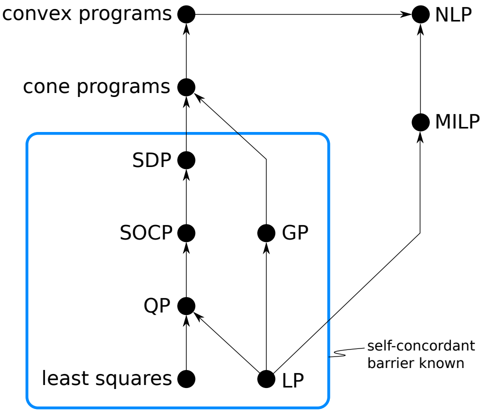
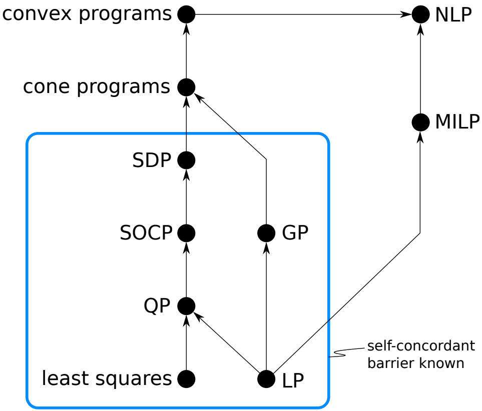


Take-aways

- Importance of mathematical structure
- Key to tractability: convexity
- Result: reliable and efficient optimization that scales to large problems

Current research interests

- Variable transformations for quasi-convex functions
- Signomial programming
- Fitting convex optimization models to data



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Alex Bayen
Karen Willcox
Mark Drela
John Hansman

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Questions

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Course notes.

Constraint sensitivities

- Consider perturbed GP:

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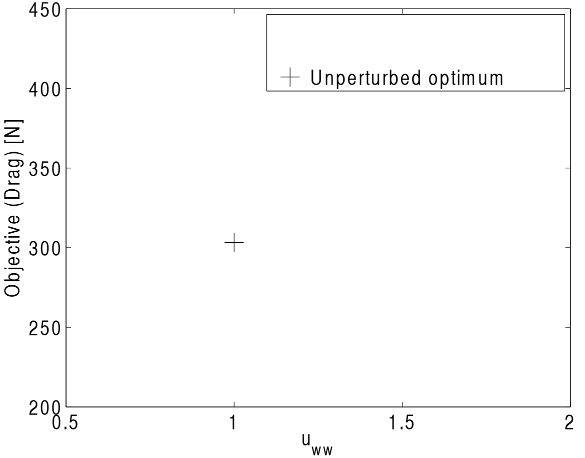
$$\left.\frac{\partial \log p^*(\boldsymbol{u})}{\partial \log u_i}\right|_{\boldsymbol{u}=\boldsymbol{1}} = \left.\frac{\partial \left(\frac{p^*(\boldsymbol{u})}{p^*(\boldsymbol{1})}\right)}{\partial \left(\frac{u_i}{1}\right)}\right|_{\boldsymbol{u}=\boldsymbol{1}} = -\lambda_i$$

Pareto frontier "linearization"



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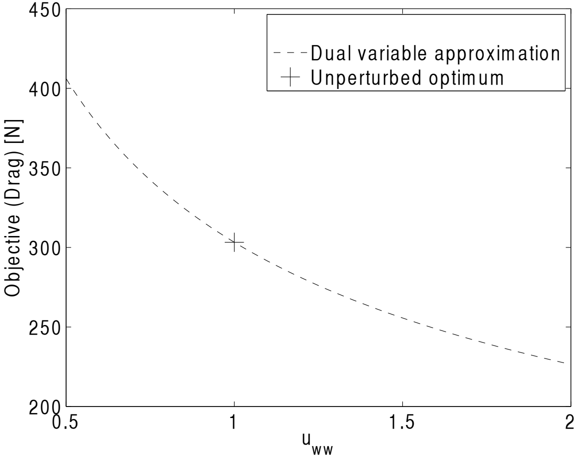

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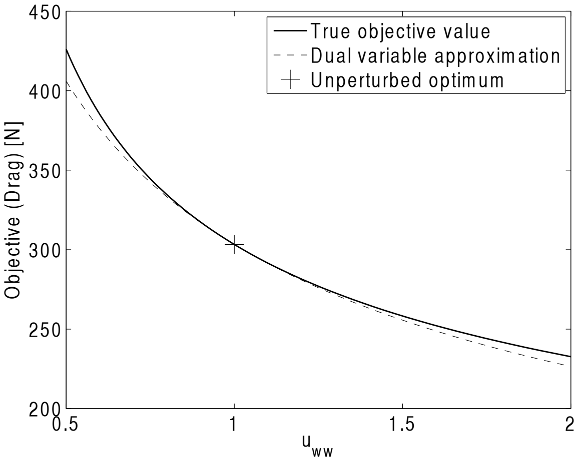

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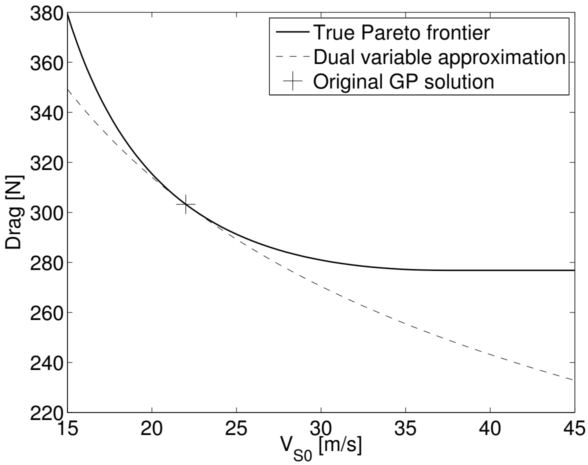

Some Useful Bounds

Dual Sensitivity Analysis

- Perturb constraints (via $\boldsymbol{u}$)
- Performance bound:

$$\log p^*(\boldsymbol{u}) \geq \log p^*(\mathbf{1}) + \boldsymbol{\lambda}^T \boldsymbol{u}$$

- An *optimistic* estimate

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Design Averaging

- Consider two designs θ_1, θ_2, with objective values p_1^*, p_2^*
- Form geometric mean design

$$\theta_3^{(i)} = \sqrt{\theta_1^{(i)} \theta_2^{(i)}}$$

- Performance bound:

$$p_3^* \leq \sqrt{p_1^* p_2^*}$$

- A *pessimistic* estimate

Posynomial Equality Relaxation

When can we guarantee $h(x) = 1$ will hold at optimum ?

$$
\begin{array}{lll}
\text{minimize} & p_0(x) & \\
\text{subject to} & p_i(x) \leq 1, & i = 1, \ldots, N_p, \\
 & m_j(x) = 1, & j = 1, \ldots, N_m \\
 & h(x) \leq 1 &
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If $\exists\ x_k$ s.t.:

- x_k does not appear in monomial equality constraints, i.e. $\frac{\partial m_j}{\partial x_k} = 0$
- p_0 monotone strictly decreasing in x_k, i.e. $\frac{\partial p_0}{\partial x_k} < 0$
- All p_i monotone decreasing in x_k, i.e. $\frac{\partial p_i}{\partial x_k} \leq 0$
- h is monotone strictly increasing in x_k, i.e. $\frac{\partial h}{\partial x_k} > 0$

$\rightarrow$ **Conditions satisfied for all relaxations presented today.**

Extensions exist for multiple $h_i(x)$, $\frac{\partial p_0}{\partial x_k} = 0$ case [Boyd et. al., 2007]

Conceptual Design – Modeling Summary

- Fuselage Pressure Loads
- Fuselage Bending Loads
- Fuselage Weight
- Steady Level Flight Relations
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- Noise
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- Wing Induced Drag
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- Interference Drags
- Airfoil Shape Optimization
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- Propulsive Efficiency
- Blade Element Momentum Theory
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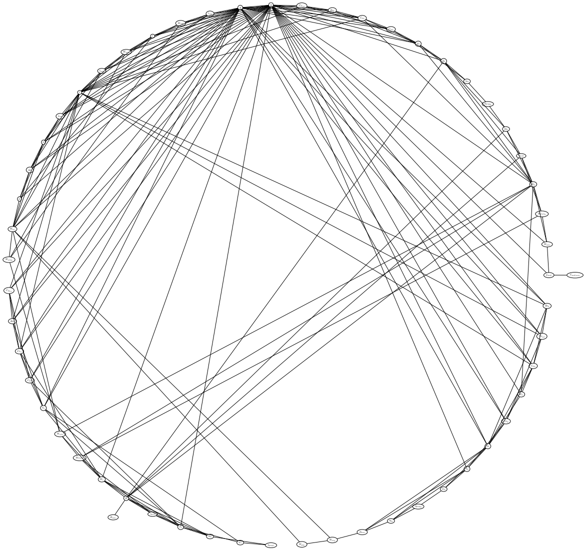
gpkit

- Open source python modeling tool
- Interfaces with MOSEK and cvxopt solvers
- Stable version release planned for November 2014
- `http://github.com/appliedopt/gpkit`

gpkit — coupling graphs



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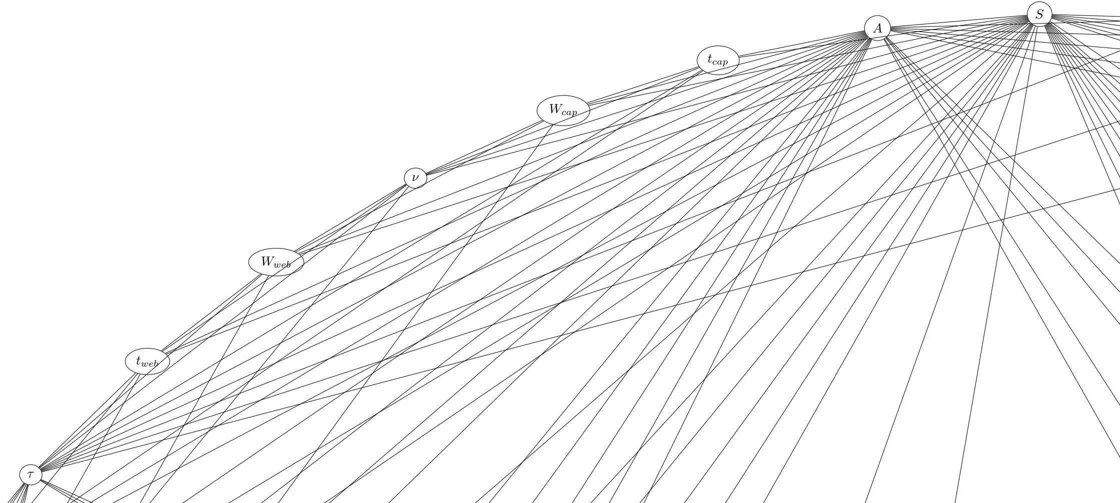

Lagrange Dual of GP

Primal problem (in convex form):

$$\begin{aligned} \text{minimize} \quad & \log \sum_{k=1}^{K_0} \exp(\boldsymbol{a}_{0k}^T \boldsymbol{y} + b_{0k}) \\ \text{subject to} \quad & \log \sum_{k=1}^{K_i} \exp(\boldsymbol{a}_{ik}^T \boldsymbol{y} + b_{ik}) \leq 0, \;\; i = 1, ..., m, \end{aligned} \tag{1}$$

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Lagrangian and dual function:

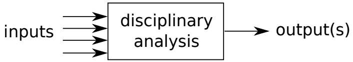
$$\begin{aligned} L(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{\lambda}, \boldsymbol{\nu}) &= \log \sum_{k=1}^{K_0} \exp z_{0k} + \sum_{i=1}^{m} \lambda_i \log \sum_{k=1}^{K_i} \exp z_{ik} + \sum_{i=0}^{m} \boldsymbol{\nu}_i^T (\boldsymbol{A}_i \boldsymbol{y} + \boldsymbol{b}_i - \boldsymbol{z}_i) \\ g(\boldsymbol{\lambda}, \boldsymbol{\nu}) &= \inf_{\boldsymbol{y}, \boldsymbol{z}} L(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{\lambda}, \boldsymbol{\nu}). \end{aligned}$$

Lagrange Dual of GP

$$
\begin{array}{ll}
\text{maximize} & \displaystyle\sum_{i=0}^{m}\left[\boldsymbol{\nu}_i^T \boldsymbol{b}_i - \sum_{k=1}^{K_i} \nu_{ik} \log \frac{\nu_{ik}}{\mathbf{1}^T \boldsymbol{\nu}_i}\right] \\
\text{subject to} & \displaystyle\sum_{i=0}^{m} \boldsymbol{\nu}_i^T \boldsymbol{A}_i = 0 \\
& \boldsymbol{\nu}_i \geq 0, \quad i = 0, \dots, m \\
& \mathbf{1}^T \boldsymbol{\nu}_0 = 1.
\end{array}
$$

- An equality-constrained entropy maximization
- (unnormalized) probability distributions $\boldsymbol{\nu}_i$ satisfy $\mathbf{1}^T \boldsymbol{\nu}_i = \lambda_i$

Fitting Reduced-Order GP-compatible Models



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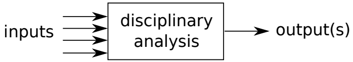

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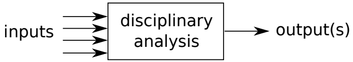

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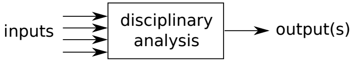

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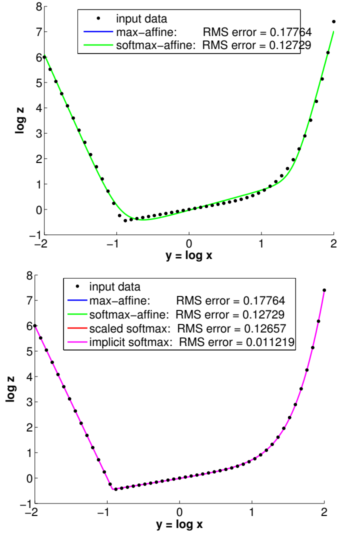

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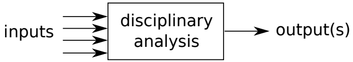

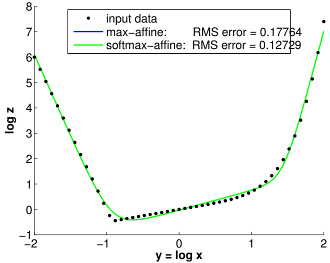

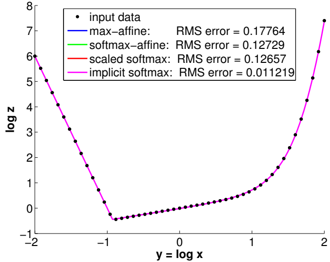

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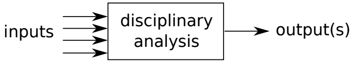

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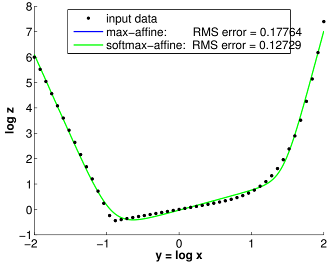

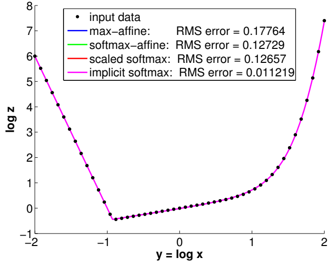

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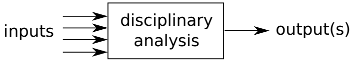

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- Many extensions, e.g. conservative fitting, sparse fitting