TOWARDS A NEW APPROACH OF MESH ADAPTATION METHODS AND ITS IMPACT ON ACCURACY FOR THE SIMULATION OF STIFF PDES

Marc Massot

The present work is the result of a team work involving

- Thomas Bellotti (CNRS, EM2C - Fédé Maths CS)
- Loïc Gouarin (École polytechnique, CMAP)
- Josselin Massot (École polytechnique, CMAP)
- Pierre Matalon (École polytechnique, CMAP)
- Laurent Séries (École polytechnique, CMAP)
- Christian Tenaud (CNRS, EM2C - Fédé Maths CS)

website of the team
https://github.com/hpc-maths/



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CONTEXT

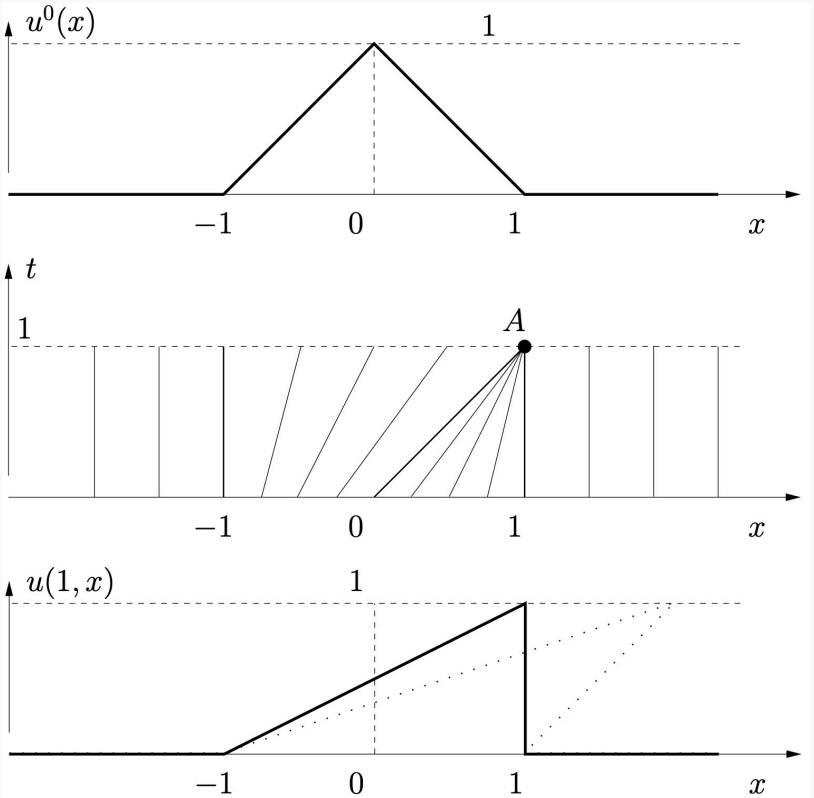
BURGERS EQUATION - SMALL HAT PROBLEM

$$\partial_t u + \partial_x (f(u)) = 0, \quad t \geq 0, \quad x \in \mathbb{R}, \qquad f(u) = \frac{u^2}{2},$$

Consider the Cauchy problem with initial cond.

$$u^0(x) = \begin{cases} 0, & x \in]-\infty, -1] \cup [1, +\infty[, \\ x+1, & x \in]-1, 0], \\ 1-x, & x \in [0, 1[. \end{cases}$$

- Shock formation at time $T^* = 1$
- Leading to irreversible solution
- RH condition governs shock dynamics



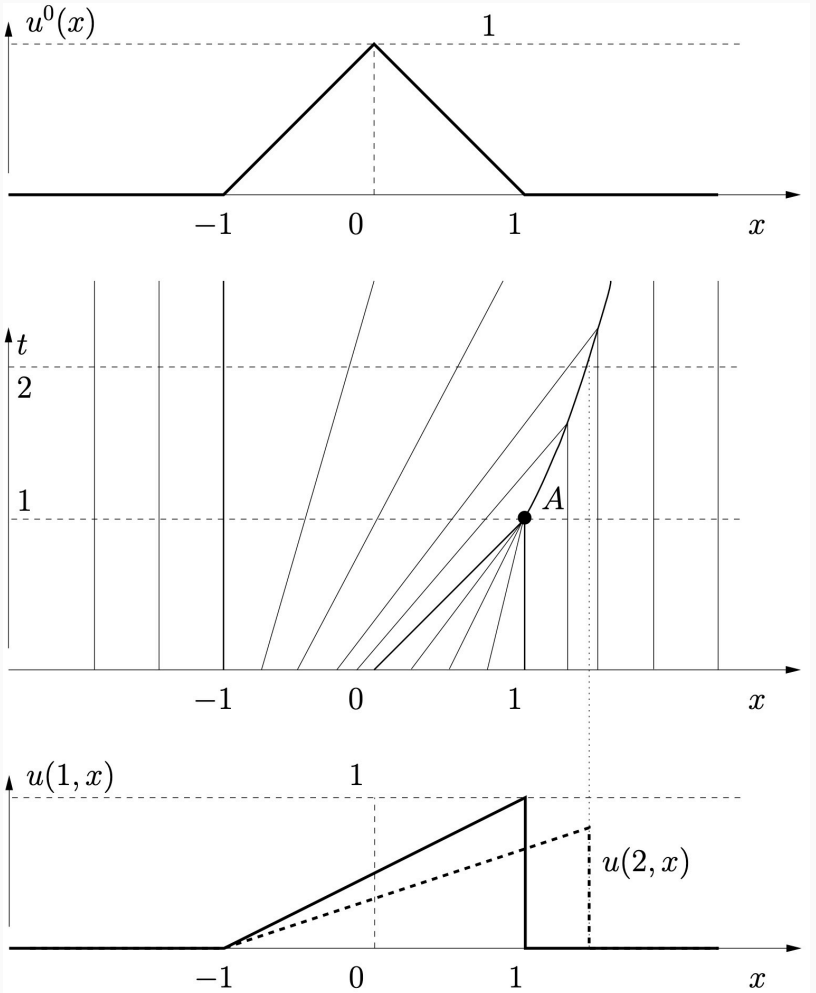
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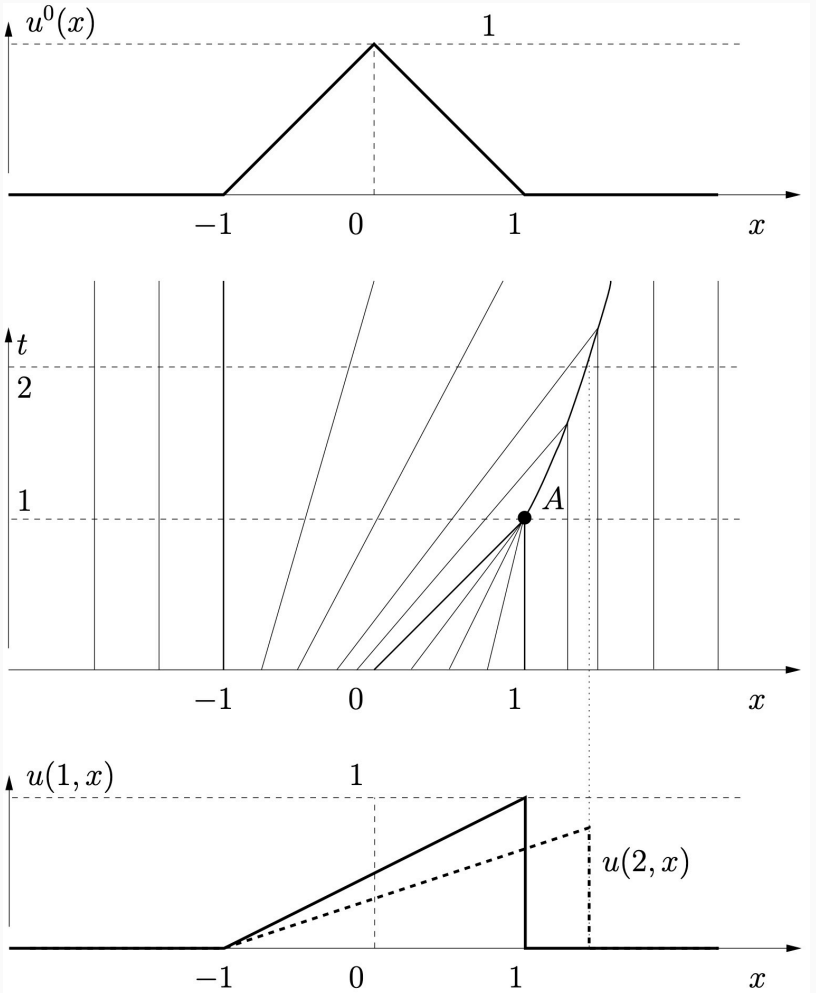
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- Shock location $\varphi(t) = \sqrt{2(1+t)} - 1$
- Propagation speed shock $\sigma(t) = 1/\sqrt{2(1+t)}$
- Shock amplitude $[u] = \sqrt{2/(1+t)}$

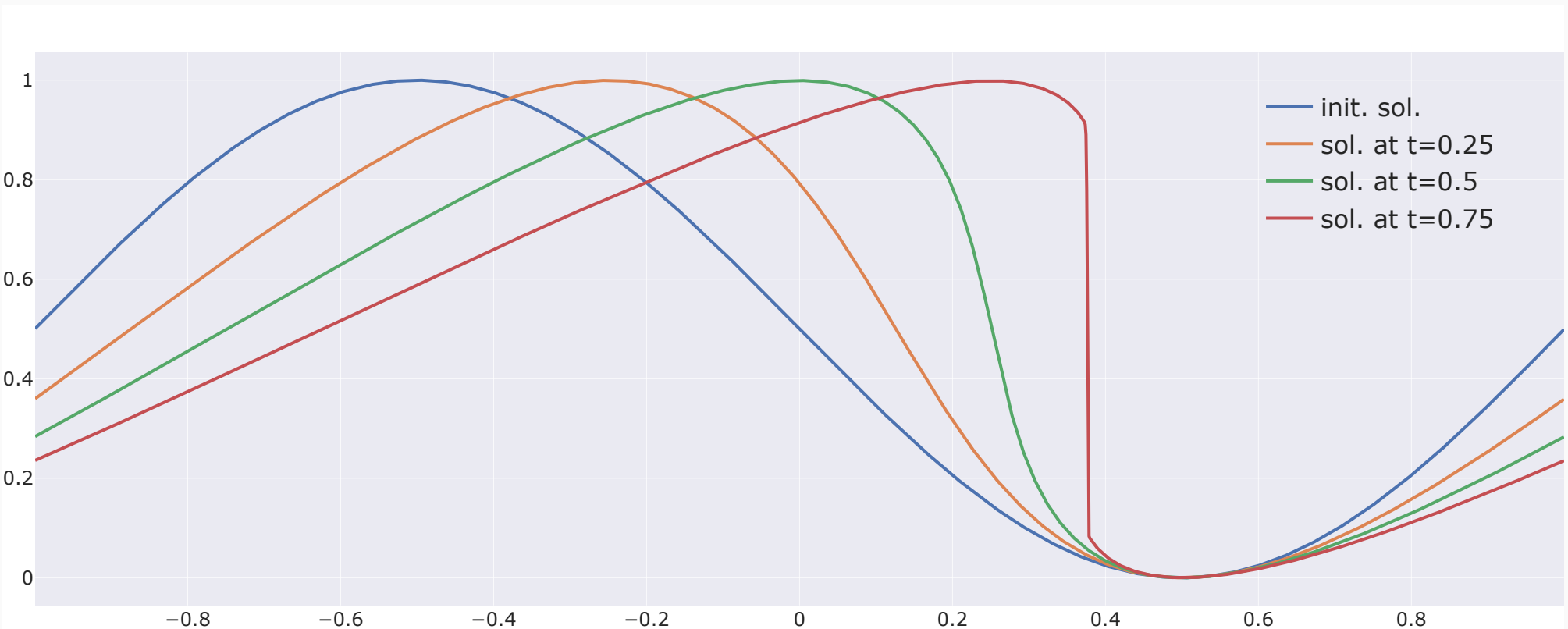


BURGERS EQUATION - SINUS PROBLEM

$$\partial_t u + \partial_x (f(u)) = 0, \quad t \geq 0, \quad x \in \mathbb{R}, \qquad f(u) = \frac{u^2}{2},$$

Consider the Cauchy problem with initial conditions:

$$u^0(x) = \frac{1}{2}(1 + \sin(\pi(x-1))) \quad x \in [-1, 1]$$

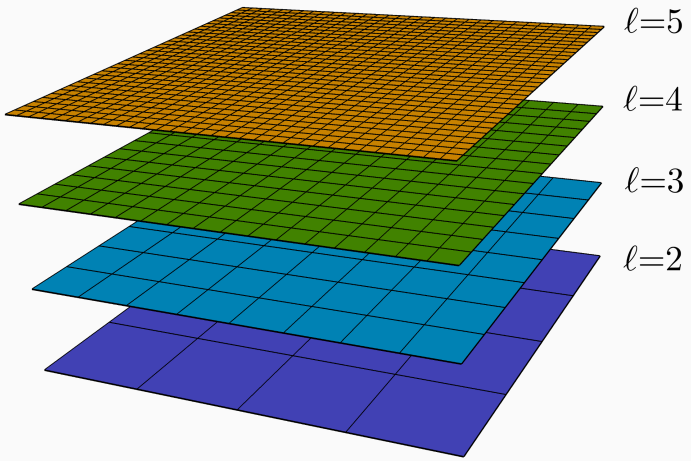


ADAPTIVE MULTIRESOLUTION

- Minimum level $\underline{\ell}$ and maximum level $\bar{\ell}$.
- Cells:

$$C_{\ell,k} := \prod_{\alpha=1}^{d} \left[2^{-\ell} k_\alpha, 2^{-\ell} (k_\alpha + 1)\right]$$

- Finest step: $\Delta x = 2^{-\bar{\ell}}$.
- Level-wise step: $\Delta x_\ell := 2^{-\ell} = 2^{\Delta\ell} \Delta x$.



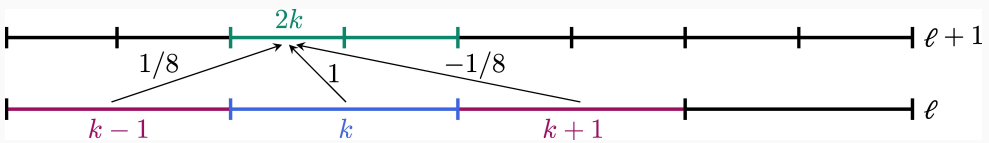
WAVELETS

Decomposition of the solution on a wavelet basis [Daubechies, '88], [Mallat, '89] to measure its local regularity. "Practical" approach by [Harten, '95], [Cohen et al., '03], [Duarte et al., '11-'16].

Projection operator

Prediction operator at order $2s+1$

$$\hat{f}_{\ell+1,2k} = f_{\ell,k} + \sum_{\sigma=1}^{s} \psi_\sigma \left(f_{\ell,k+\sigma} - f_{\ell,k-\sigma} \right)$$



Details are **regularity indicator**

$$\mathrm{d}_{\ell,k} := f_{\ell,k} - \hat{f}_{\ell,k}$$

Let $f \in W^{\nu,\infty}$ (neigh. of $C_{\ell,k}$), then

$$|\mathrm{d}_{\ell,k}| \lesssim 2^{-\ell \min(\nu,2s+1)} |f|_{W^{\min(\nu,2s+1),\infty}}$$

Fast wavelet transform:

means at the finest level can be recast as means at the coarsest level + details

$$f_{\bar{\ell}} \iff \left(f_{\underline{\ell}}, d_{\underline{\ell}+1}, \ldots, d_{\bar{\ell}} \right)$$

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MESH COARSENING (STATIC)

Local regularity of the solution allows to select areas to coarsen

$$f_{\bar{\ell}} \to \left(f_{\underline{\ell}}, \mathbf{d}_{\underline{\ell}+1}, \dots, \mathbf{d}_{\bar{\ell}}\right) \to \left(f_{\underline{\ell}}, \tilde{\mathbf{d}}_{\underline{\ell}+1}, \dots, \tilde{\mathbf{d}}_{\bar{\ell}}\right) \to \tilde{f}_{\bar{\ell}}$$

$$\tilde{\mathrm{d}}_{\ell,k} = \begin{cases} 0, & \text{if } |\mathbf{d}_{\ell,k}| \le \epsilon_\ell = 2^{-d\Delta\ell}\epsilon, \\ \mathrm{d}_{\ell,k}, & \text{otherwise} \end{cases} \quad \to \quad \left\|\mathbf{f}_{\bar{\ell}} - \tilde{\mathbf{f}}_{\bar{\ell}}\right\|_{\ell^p} \lesssim \epsilon$$

Set a small (below ϵ_ℓ) detail to zero $\equiv$ erase the cell $C_{\ell,k}$ from the structure

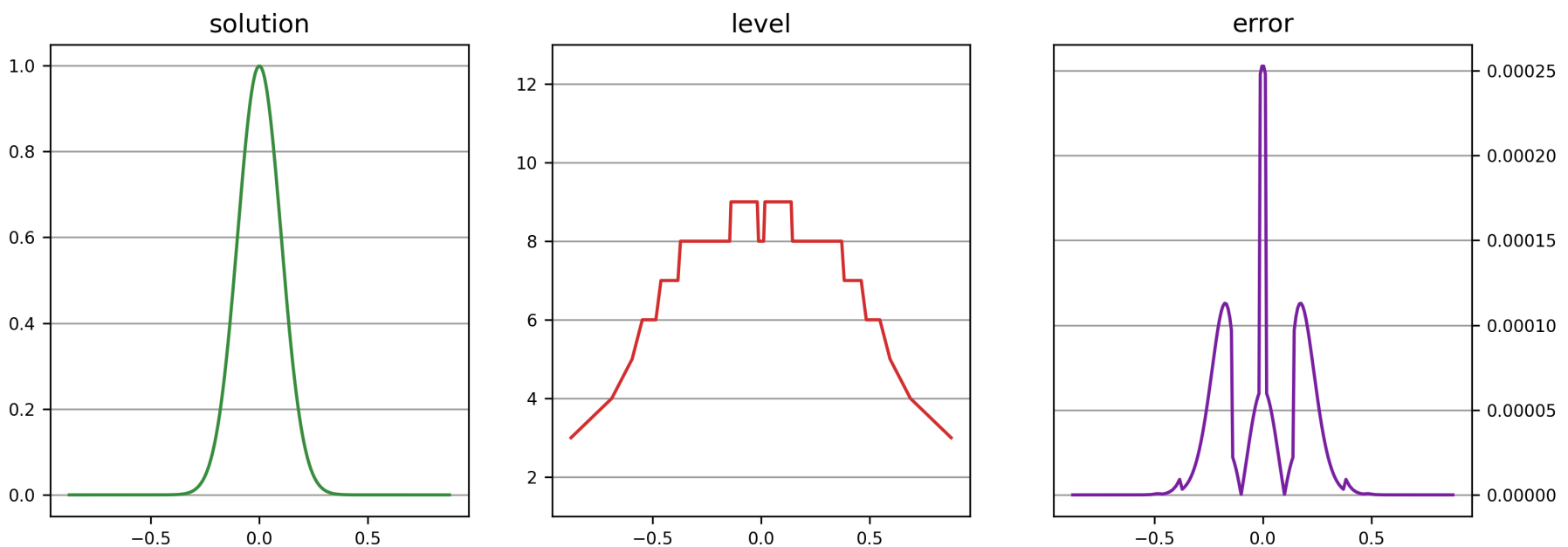
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EXAMPLES

Equation

$$f(x) = exp(-50x^2) \text{ for } x \in [-1, 1]$$

min level	1
max level	12
ε	10^{-3}
compression rate	96.29%
error	0.00078

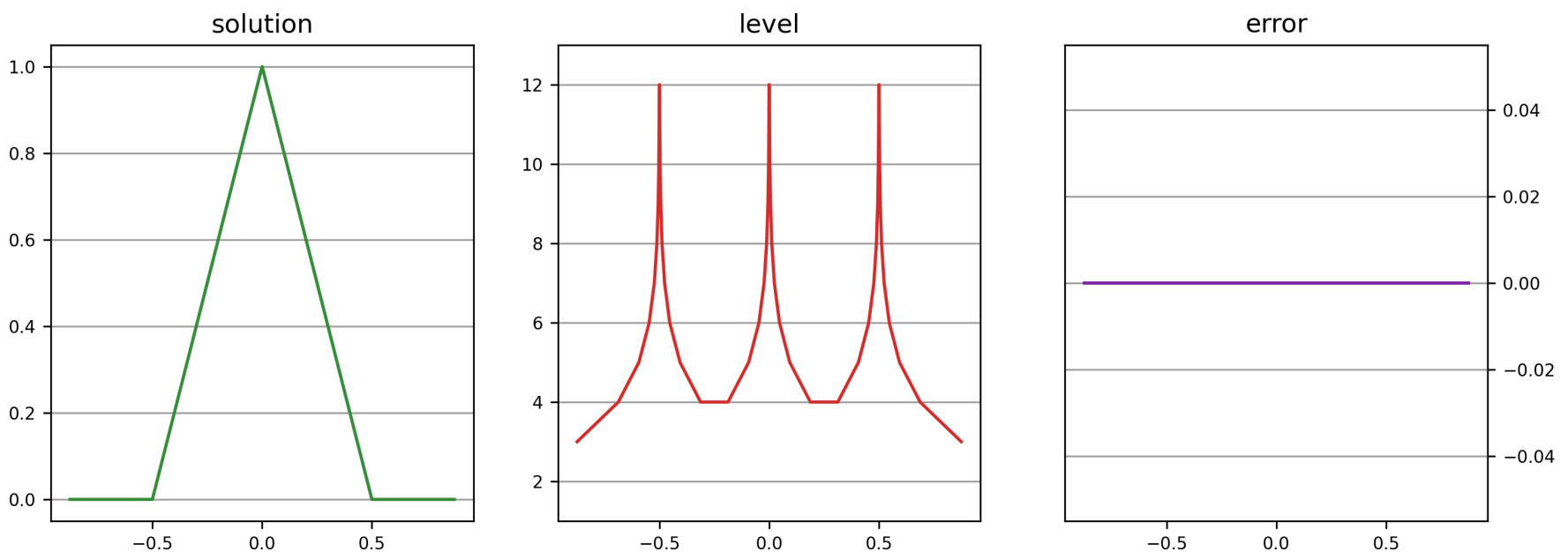


EXAMPLES

Equation

$$f(x) = \begin{cases} 1 - |2x| \text{ if } -0.5 < x < 0.5, \\ 0 \text{ elsewhere} \end{cases}$$

min level	1
max level	12
ε	10^{-3}
compression rate	98.49%
error	0

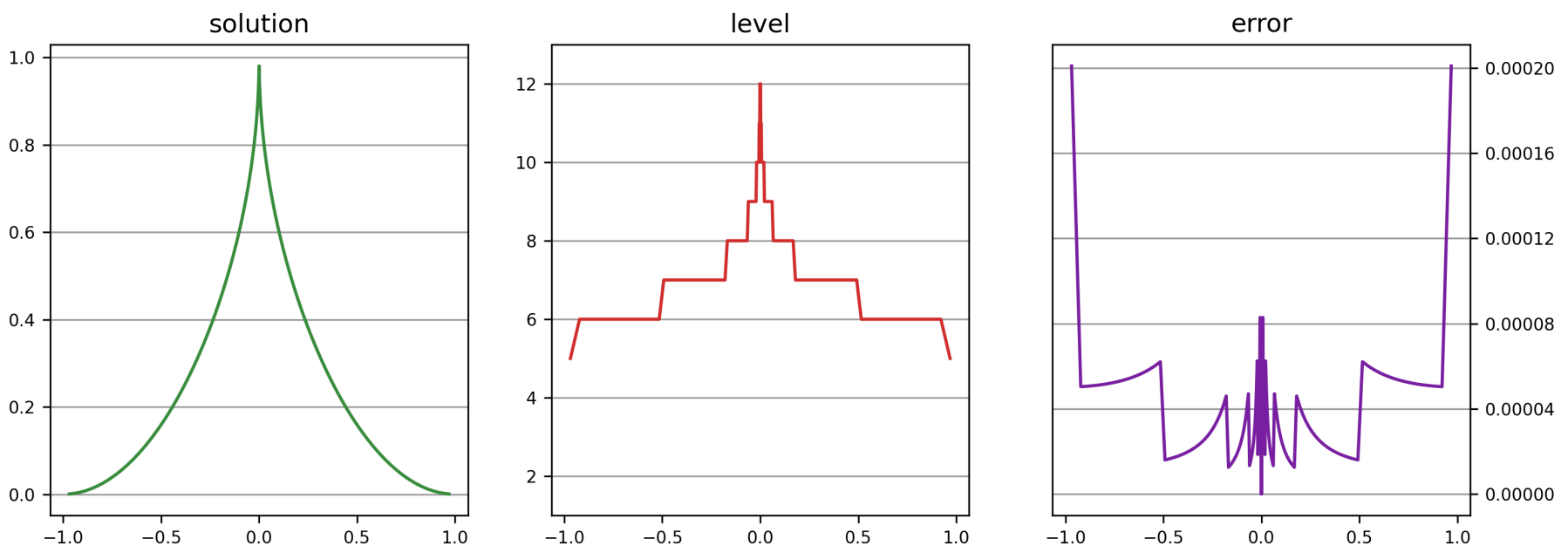


EXAMPLES

Equation

$$f(x) = 1 - \sqrt{\left|sin\left(\frac{\pi}{2}x\right)\right|} \text{ for } x \in [-1, 1]$$

min level	1
max level	12
ε	10^{-3}
compression rate	96.29%
error	0.00053

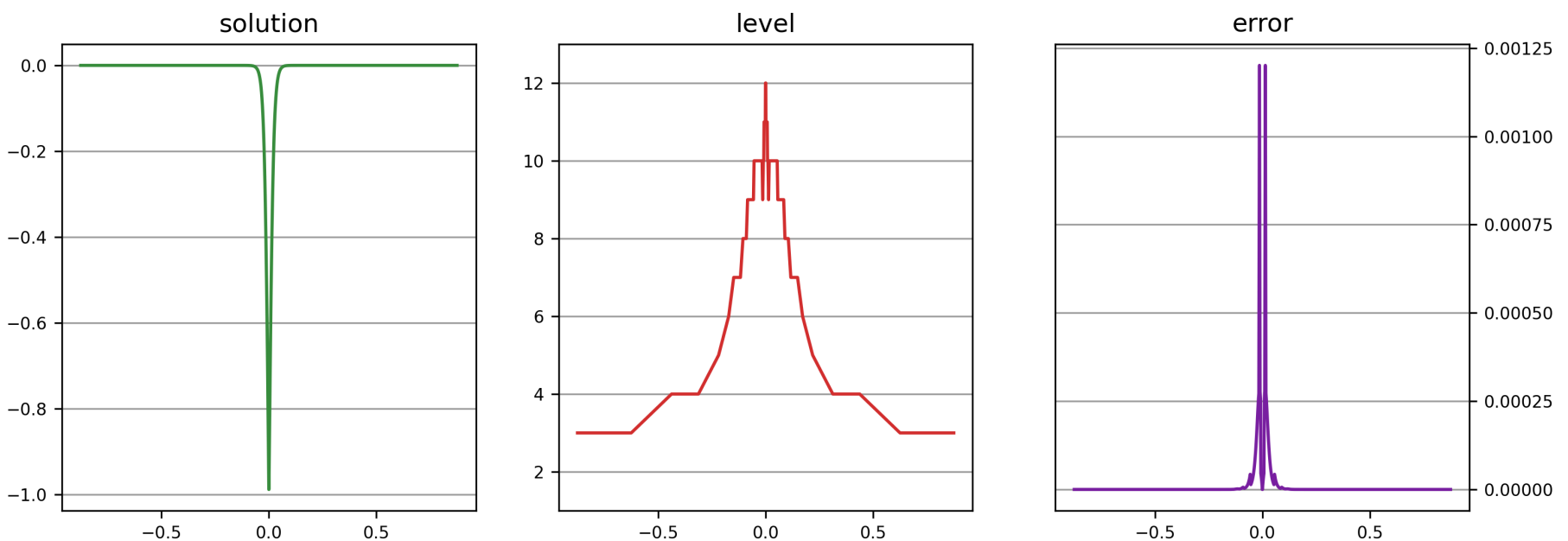


EXAMPLES

Equation

$$f(x) = \tanh(50|x|) - 1 \text{ for } x \in [-1, 1]$$

min level	1
max level	12
ε	10^{-3}
compression rate	97.46%
error	0.002



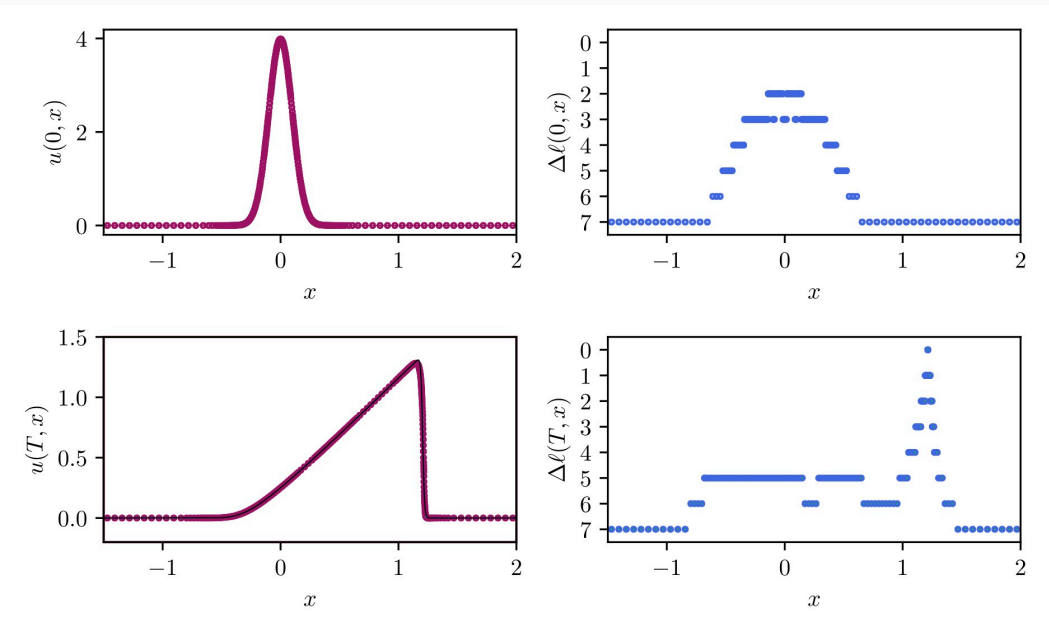
TIME EVOLUTION OF PDES

- Finite volumes with global time step $\Delta t = \Lambda(\Delta x)$
- Use dynamic mesh refinement

Mesh updated using "old" information at time t to accommodate the one at time $t + \Delta t$

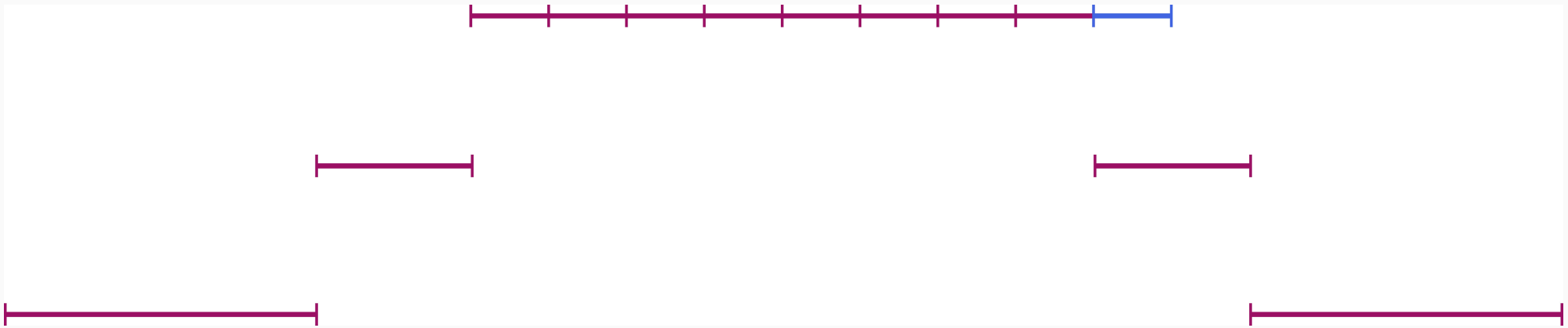
- Propagation of information : add security cells
- Formation of singularities : (regularity index: $\nu = 0$, $\mu = \min(\nu, 2s + 1)$) refine if

$$|\mathbf{d}_{\ell,k}| \geq \epsilon_\ell \, 2^{d+\mu}$$

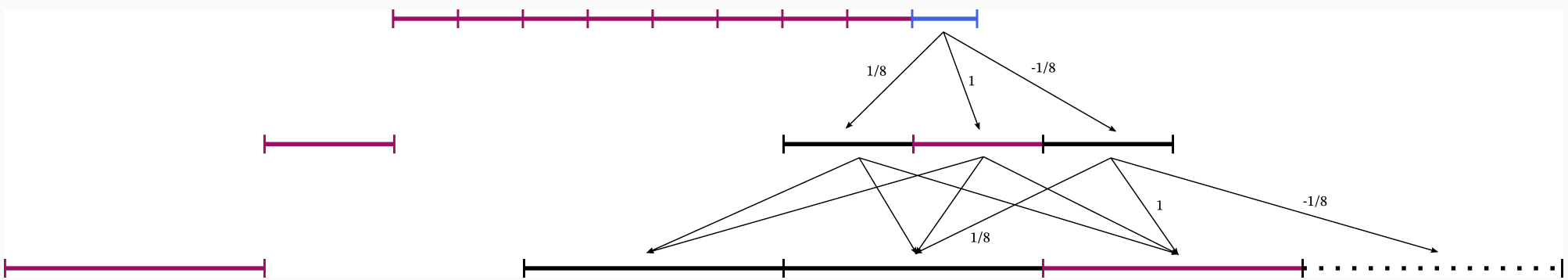


FINITE VOLUMES / CONSERVATION / ORDER

Flux evaluation at interfaces between levels

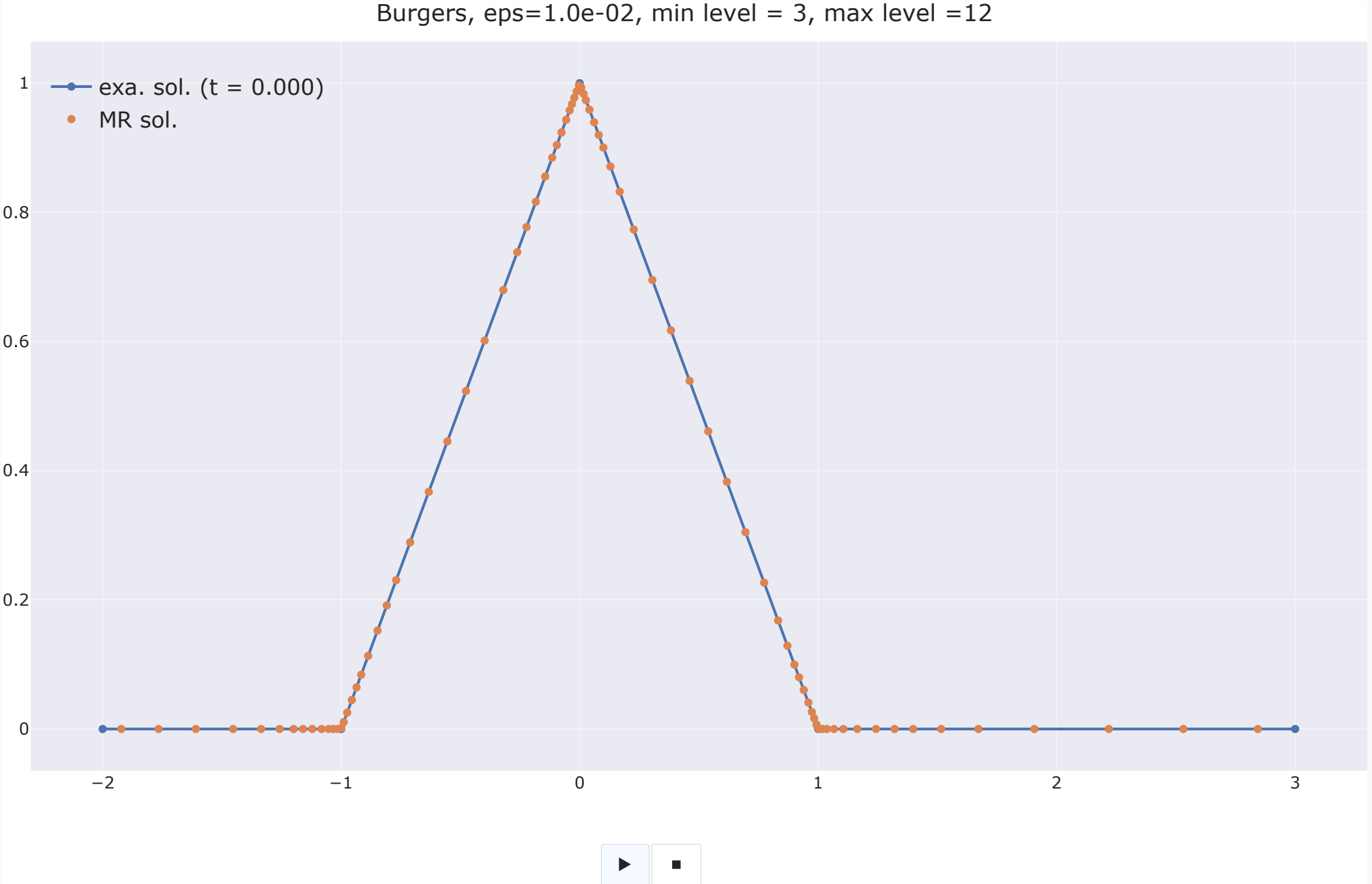


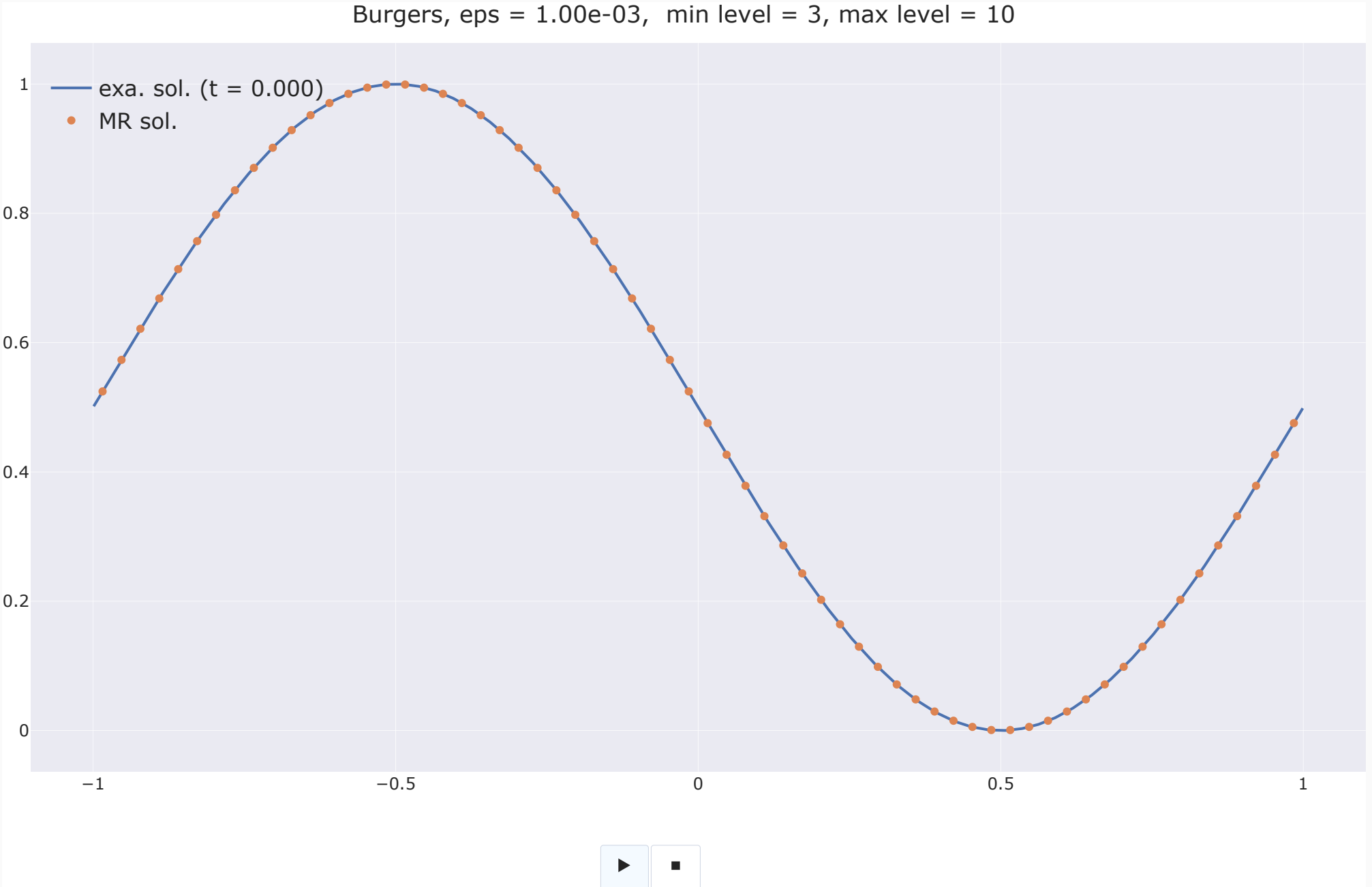
Using the prediction operator allows to evaluate fluxes at the same level



Finite volume method

We use a Godunov flux for the small hat problem


Burgers, eps=1.0e-02, min level = 3, max level =12
exa. sol. (t = 0.000)
MR sol.
1
0.8
0.6
0.4
0.2
0
−2
−1
0
1
2
3


Burgers, eps = 1.00e-03, min level = 3, max level = 10
exa. sol. (t = 0.000)
MR sol.
1
0.8
0.6
0.4
0.2
0
−1
−0.5
0
0.5
1

NUMERICAL ANALYSIS AND MODIFIED EQUATIONS

LINEAR SCALAR TRANSPORT EQUATION

In this work, we are concerned with the numerical solution of the Cauchy problem associated with the linear scalar conservation law

$$\partial_t u(t,x) + V \partial_x u(t,x) = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}$$

where V is the transport velocity, taken $V > 0$ without loss of generality. we consider 1 d problems. The extension to $2\,\mathrm{d}$ / $3\,\mathrm{d}$ problems is straightforward and usually done by tensorization [Bellotti2022] and yields analogous conclusions. The discrete volumes are

$$C_{\ell,k} := \left[2^{-\ell} k, 2^{-\ell}(k+1)\right], \quad k \in \{0, 2^\ell - 1\},$$

for any $\ell \in \{\underline{\ell}, \bar{\ell}\}$. The measure of each cell at level ℓ is $\Delta x_\ell := 2^{-\ell}$ and we shall indicate $\Delta x := \Delta x_{\bar{\ell}}$. The cell centers are $x_{\ell,k} := 2^{-\ell}(k + 1/2)$. Finally, we shall indicate $\Delta\ell := \bar{\ell} - \ell$, hence $\Delta x_\ell = 2^{\Delta\ell} \Delta x$.

FINITE VOLUME SCHEME

Finite Volume scheme at the finest level of resolution $\bar{\ell}$ for any cell of indices $\bar{k} \in \{0, 2^{\bar{\ell}} - 1\}$. Explicit schemes read:

$$\mathrm{v}_{\bar{\ell},\bar{k}}^{n+1} = \mathrm{v}_{\bar{\ell},\bar{k}}^{n} - \frac{\Delta t}{\Delta x}\left(\Phi\left(\mathrm{v}_{\bar{\ell},\bar{k}+1/2}^{n}\right) - \Phi\left(\mathrm{v}_{\bar{\ell},\bar{k}-1/2}^{n}\right)\right)$$

where we utilize the same linear numerical flux for the left and the right flux (conservativity)

$$\Phi\left(\mathbf{v}_{\bar{\ell},\bar{k}-1/2}\right) := V \sum_{\alpha=\underline{\alpha}}^{\bar{\alpha}} \phi_\alpha \mathbf{v}_{\bar{\ell},\bar{k}+\alpha}, \quad \Phi\left(\mathbf{v}_{\bar{\ell},\bar{k}+1/2}\right) := V \sum_{\alpha=\underline{\alpha}}^{\bar{\alpha}} \phi_\alpha \upsilon_{\bar{\ell},\bar{k}+1+\alpha}$$

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MODIFIED EQUATIONS

[Carpentier et al 97] or Cauchy-Kowalewski procedure [Harten et al 87]

$$\partial_t u\left(t^n, x_{\bar{\ell},\bar{k}}\right) + V\partial_x u\left(t^n, x_{\bar{\ell},\bar{k}}\right) = \sum_{h=2}^{+\infty} \Delta x^{h-1} \sigma_h \partial_x^h u\left(t^n, x_{\bar{\ell},\bar{k}}\right)$$

- Upwind scheme

$$\partial_t u + V\partial_x u = \frac{\Delta x V}{2}(1 - \lambda V)\partial_{xx} u + O\left(\Delta x^2\right)$$

- Lax-Wendroff scheme

$$\partial_t u + V\partial_x u = -\frac{\Delta x^2 V}{6}\left(1 - \lambda^2 V^2\right)\partial_x^3 u + O\left(\Delta x^3\right)$$

- OSMP-3 scheme

$$\partial_t u + V\partial_x u = \frac{\Delta x^3 V}{24}\left(-\lambda^3 V^2 + 2\lambda^2 V^2 + \lambda V - 2\right)\partial_x^4 u + O\left(\Delta x^4\right)$$

HOW TO INCLUDE MRA I

We introduce the reconstruction operator $\hat{s}$ instead of s on the cells $\left(\bar{\ell}, 2^{\Delta\ell} k + \delta\right)$ for any $\delta \in \mathbb{Z}$ at the finest level

- $\hat{s} = s$: exact local flux reconstruction [Cohen et al. 2003].
- $\hat{s} = 0$ but $s > 0$, direct evaluation or naive evaluation [Hovhannisyan et al 2010].

$$\mathbf{w}_{\bar{\ell},\bar{k}}^{n+1} = \mathbf{w}_{\bar{\ell},\bar{k}}^{n} - \frac{\Delta t}{\Delta x}\left(\Phi\left(\hat{\mathbf{w}}_{\bar{\ell},\bar{k}+1/2}^{n}\right) - \Phi\left(\hat{\mathbf{w}}_{\bar{\ell},\bar{k}-1/2}^{n}\right)\right)$$

$$\Phi\left(\hat{\mathbf{w}}_{\bar{\ell},\bar{k}-1/2}\right) := V \sum_{\alpha=\underline{\alpha}}^{\bar{\alpha}} \phi_\alpha \hat{\mathbf{w}}_{\bar{\ell},\bar{k}+\alpha}$$

HOW TO INCLUDE MRA II

Let now $(\ell, k) \in S\left(\tilde{\Lambda}^{n+1}\right)$, taking the projection yields the multiresolution scheme

$$\mathbf{w}_{\ell,k}^{n+1} = \mathbf{w}_{\ell,k}^{n} - \frac{\Delta t}{\Delta x_\ell}\left(\Phi\left(\hat{\hat{\mathbf{w}}}^{n}_{\bar{\ell},2^{\Delta\ell}(k+1)+1/2}\right) - \Phi\left(\hat{\hat{\mathbf{w}}}^{n}_{\bar{\ell},2^{\Delta\ell}k-1/2}\right)\right)$$

$$\Phi\left(\hat{\hat{\mathbf{w}}}_{\bar{\ell},2^{\Delta\ell}k-1/2}\right) := V\sum_{\alpha=\underline{\alpha}}^{\bar{\alpha}} \phi_\alpha \hat{\hat{\mathbf{w}}}_{\bar{\ell},2^{\Delta\ell}k+\alpha}$$

Some information is loss because of the averaging procedure: two different schemes we can consider for the computation of the modified equations.

Theorem

The local truncation error of the reference Finite Volume scheme and the one of the adaptive Finite Volume scheme are the same up to order $2\hat{s}+1$ included.

MODIFIED EQUATIONS INCLUDING MRA

This result establishes at which order the modified equations of the reference scheme are perturbed by the introduction of the adaptive scheme. However, it does not characterize the terms in the modified equations above order $2\hat{s}+1$ in Δx (symbolic computations).

- Upwind scheme (original)

$$\partial_t u + V\partial_x u = \frac{\Delta x V}{2}(1-\lambda V)\partial_{xx} u + O\left(\Delta x^2\right)$$

- Upwind scheme with MRA

$$\partial_t u + V\partial_x u = \frac{\Delta x V}{2}\left(2^{\Delta \ell} - \lambda V\right)\partial_{xx} u + O\left(\Delta x^2\right), \qquad \text{for } \hat{s}=0$$

$$\partial_t u + V\partial_x u = \frac{\Delta x V}{2}(1-\lambda V)\partial_{xx} u - \frac{\Delta x^2 V}{6}\left(1-\lambda^2 V^2\right)\partial_x^3 u + \frac{\Delta x^3 V}{24}\left(-3\Delta\ell 2^{2\Delta\ell} + 2^{2\Delta\ell} - \lambda^3 V^3\right)\partial_x^4 u + O\left(\Delta x^4\right), \qquad \text{for } \hat{s}=1$$

MODIFIED EQUATIONS INCLUDING MRA

This result establishes at which order the modified equations of the reference scheme are perturbed by the introduction of the adaptive scheme. However, it does not characterize the terms in the modified equations above order $2\hat{s}+1$ in Δx (symbolic computations).

- Lax-Wendroff scheme (original)

$$\partial_t u + V\partial_x u = -\frac{\Delta x^2 V}{6}\left(1-\lambda^2 V^2\right)\partial_x^3 u + O\left(\Delta x^3\right)$$

- Lax-Wendroff scheme with MRA

$$\partial_t u + V\partial_x u = \frac{\Delta x \lambda V^2}{2}\left(2^{\Delta\ell}-1\right)\partial_{xx} u + O\left(\Delta x^2\right), \qquad \text{for } \hat{s}=0$$

$$\partial_t u + V\partial_x u = -\frac{\Delta x^2 V}{6}\left(1-\lambda^2 V^2\right)\partial_x^3 u + \frac{\Delta x^3 \lambda V^2}{24}\left(-3\Delta\ell 2^{2\Delta\ell} + 2^{2\Delta\ell} - \lambda^2 V^2\right)\partial_x^4 u + O\left(\Delta x^4\right), \qquad \text{for } \hat{s}=1$$

THEORETICAL RESULTS ON THE GLOBAL ERROR

Theorem 2

Assume that

- The reference scheme satisfies the restricted stability condition $\|E\| \leq 1$
- The Harten-like scheme satisfies the restricted stability condition $\left\|\bar{E}_\Lambda\right\| \leq 1$ for any Λ.

Then, for smooth solution, in the limit $\Delta x \to 0$ (i.e. $\bar{\ell} \to +\infty$) and for $\Delta \underline{\ell} = \bar{\ell} - \underline{\ell}$ kept fixed, we have the error estimate

$$\left\|\mathbf{v}_{\bar{\ell}}^n - \mathbf{w}_{\bar{\ell}}^n\right\| \leq C_{tr} t^n \Delta x^{2\hat{s}+1} + C_{mr} \frac{t^n}{\lambda \Delta x} \epsilon$$

where $C_{tr} = C_{tr}\left(\bar{\ell} - \underline{\ell}, (\phi_\alpha)_\alpha, \lambda, \hat{s}, V\right)$ and $C_{mr} = C_{mr}\left(\bar{\ell} - \underline{\ell}, (\phi_\alpha)_\alpha, \lambda, \hat{s}, s, V\right)$.

$$\left\|\mathbf{u}_{\bar{\ell}}^n - \mathbf{w}_{\bar{\ell}}^n\right\| \leq C_{ref} t^n \Delta x^\theta + C_{tr} t^n \Delta x^{2\hat{s}+1} + C_{mr} \frac{t^n}{\lambda \Delta x} \epsilon$$

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COMMENTS ON THEOREM

- The error estimate contains three contributions: the **discretization error** of the reference scheme, the **perturbation error** between the reference and the adaptive scheme, and the **thresholding error** coming from the multiresolution
- The constant C_{tr} generally grows exponentially with $\bar{\ell} - \underline{\ell}$, sometimes also involving linear terms, i.e. $\hat{s} = 1$. We have the following cases:
  - $\theta < 2\hat{s} + 1$. The error of the reference scheme dominates the perturbation introduced by the adaptive scheme $\left\| \mathbf{u}_{\bar{\ell}}^N - \mathbf{w}_{\bar{\ell}}^N \right\| \leq C_{\text{ref}} T \Delta x^{\theta} + C_{\text{mr}} \frac{T}{\lambda \Delta x} \epsilon$. A thresholding error of the same order as the reference error $\epsilon \sim \Delta x^{\theta+1}$.
  - $\theta = 2\hat{s} + 1$. The error of the reference scheme and the perturbation order are comparable (**first example!**) We have $\left\| \mathbf{u}_{\bar{\ell}}^N - \mathbf{w}_{\bar{\ell}}^N \right\| \leq (C_{\text{ref}} + C_{\text{tr}}) T \Delta x^{\theta} + C_{\text{mr}} \frac{T}{\lambda \Delta x} \epsilon$.
  - $\theta > 2\hat{s} + 1$. The perturbation introduced by the adaptive scheme dominates the error of the reference scheme. Therefore, multiresolution introduces a large perturbation that yields a different convergence rate. We have $\left\| \mathbf{u}_{\bar{\ell}}^N - \mathbf{w}_{\bar{\ell}}^N \right\| \leq C_{\text{tr}} T \Delta x^{2\hat{s}+1} + C_{\text{mr}} \frac{T}{\lambda \Delta x} \epsilon$, thus $\epsilon \sim \Delta x^{2\hat{s}+2}$ (**AMR !**)

HOW TO COMPUTE FLUXES AT THE FINEST LEVEL



HOW TO COMPUTE FLUXES AT THE FINEST LEVEL



HOW TO COMPUTE FLUXES AT THE FINEST LEVEL

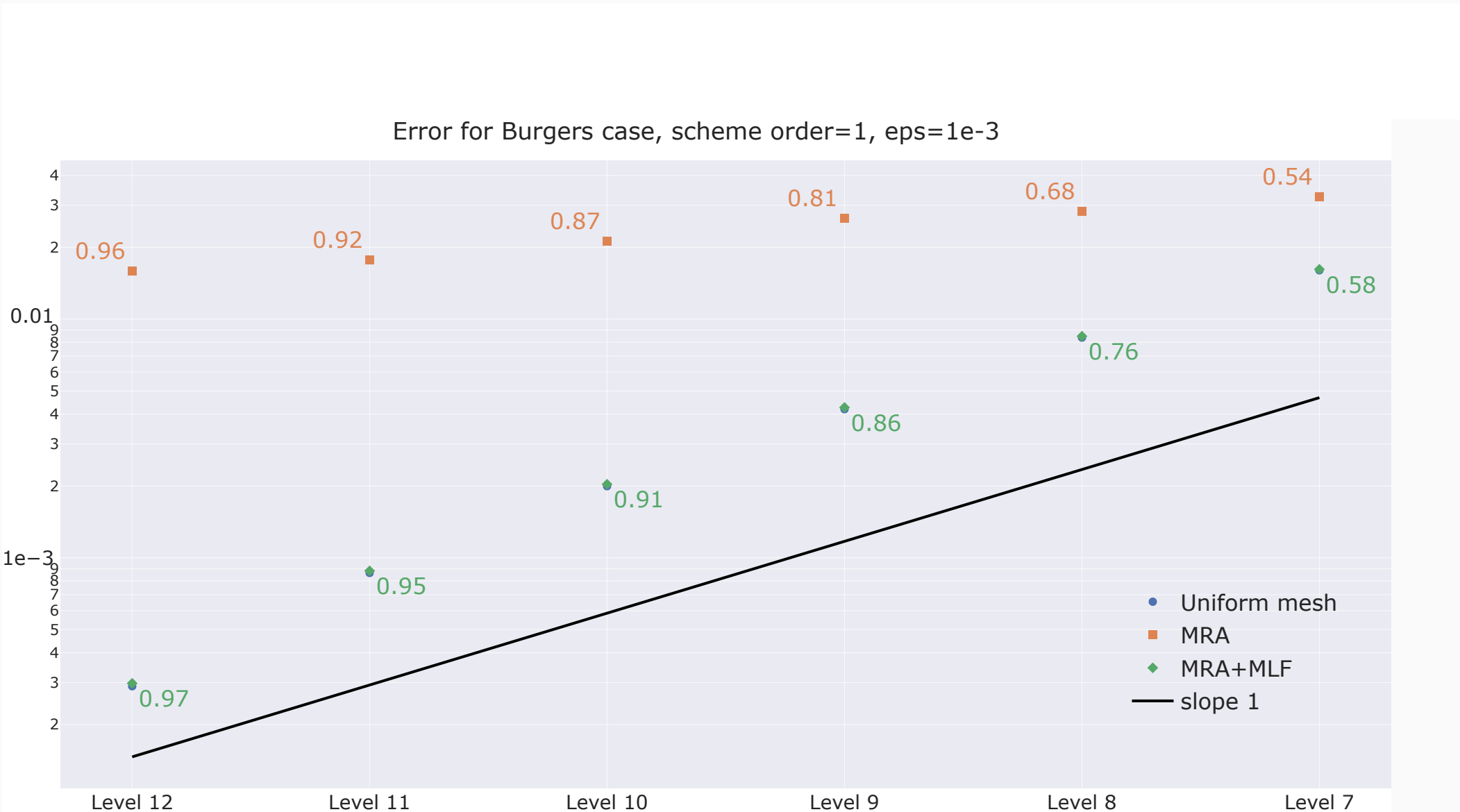


HOW TO COMPUTE FLUXES AT THE FINEST LEVEL

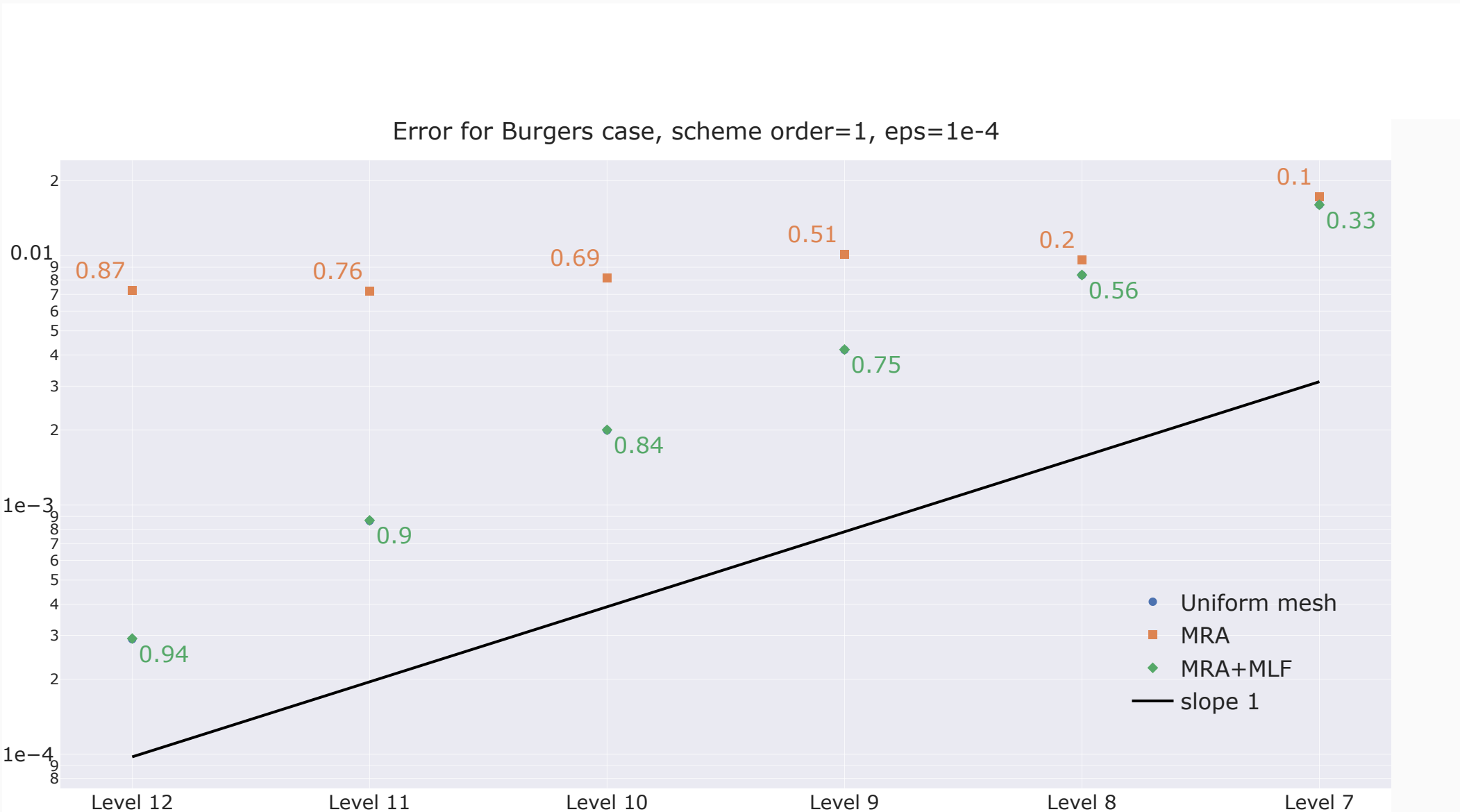


BURGERS RESULTS

BURGERS RESULTS (ERROR FOR SCHEME ORDER 1)



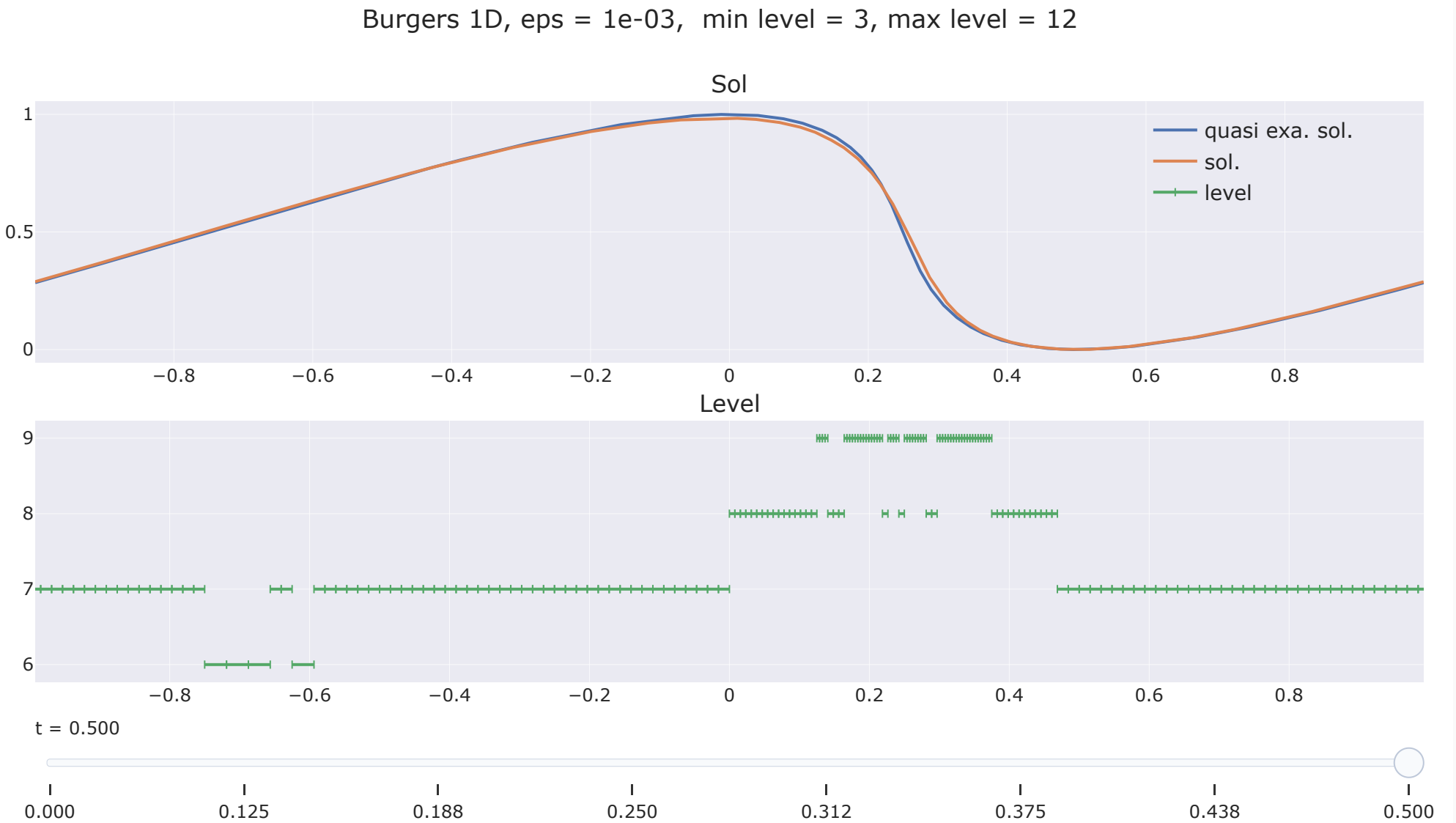
BURGERS RESULTS (ERROR FOR SCHEME ORDER 1)



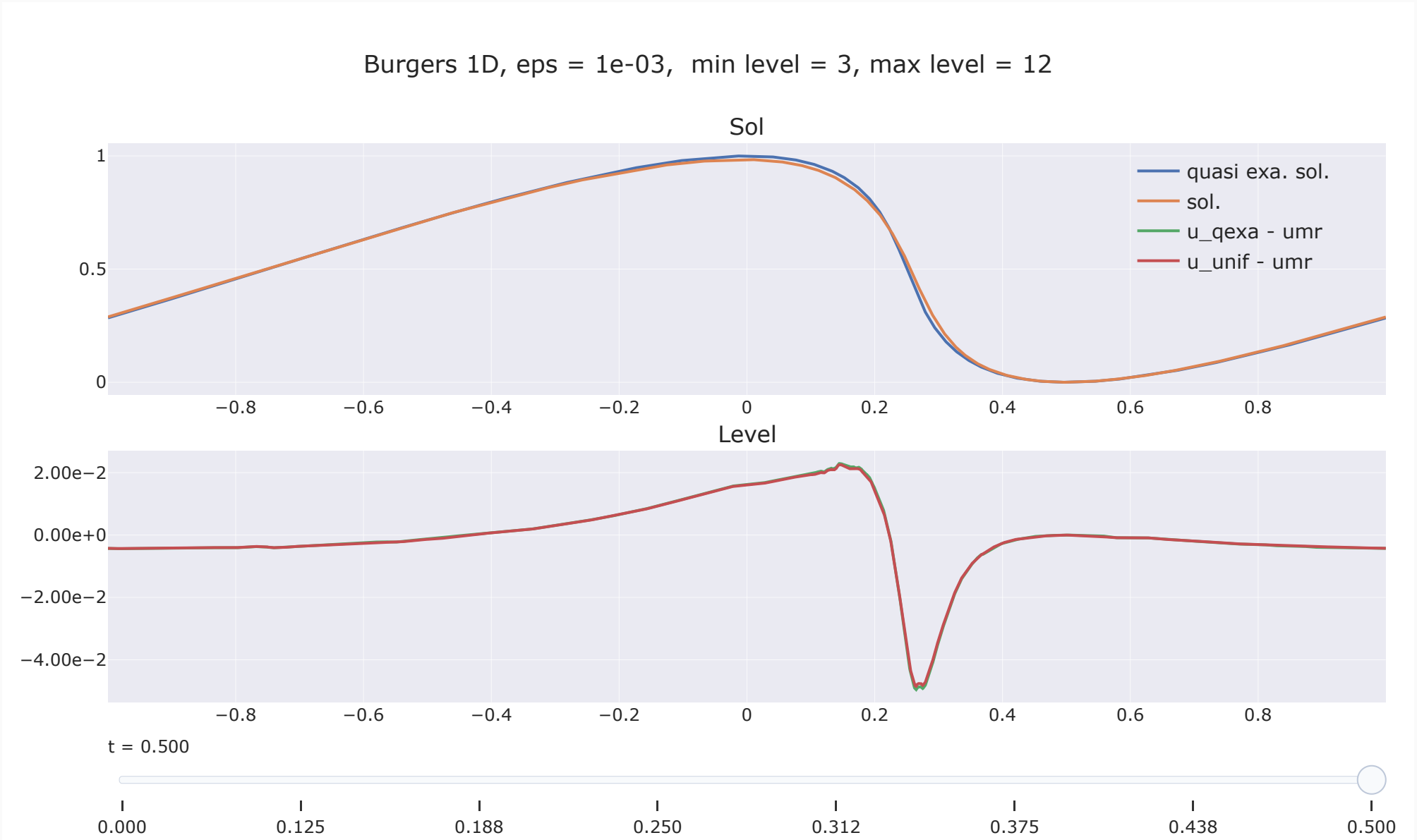
BURGERS RESULTS (ERROR FOR SCHEME ORDER 1)



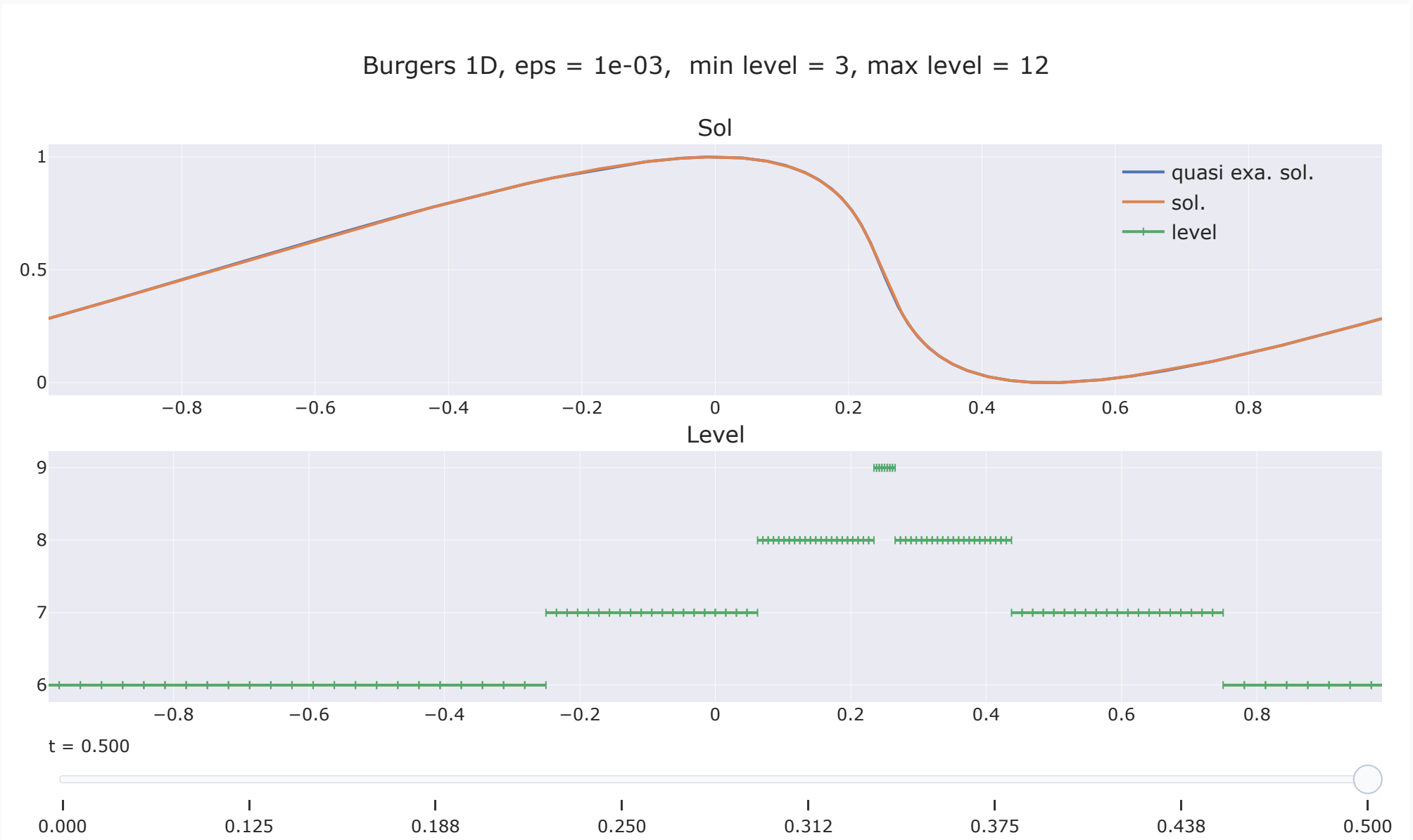
BURGERS RESULTS (MR SOLUTION AND LEVEL FOR SCHEME ORDER 1)



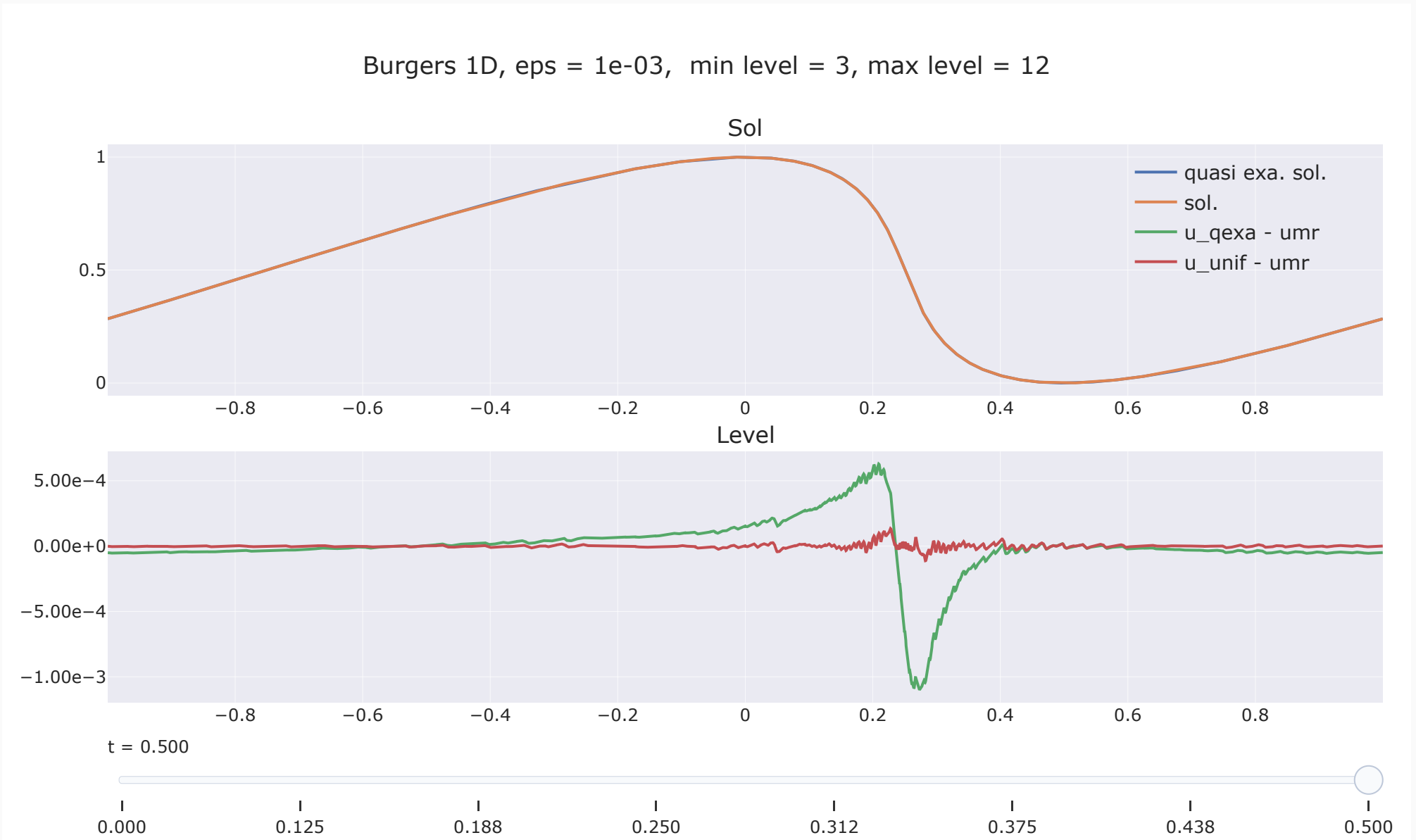
BURGERS RESULTS (MR SOLUTION AND ERROR FOR SCHEME ORDER 1)



BURGERS RESULTS (MR+MLF SOL. AND LEVEL FOR SCHEME ORDER 1)



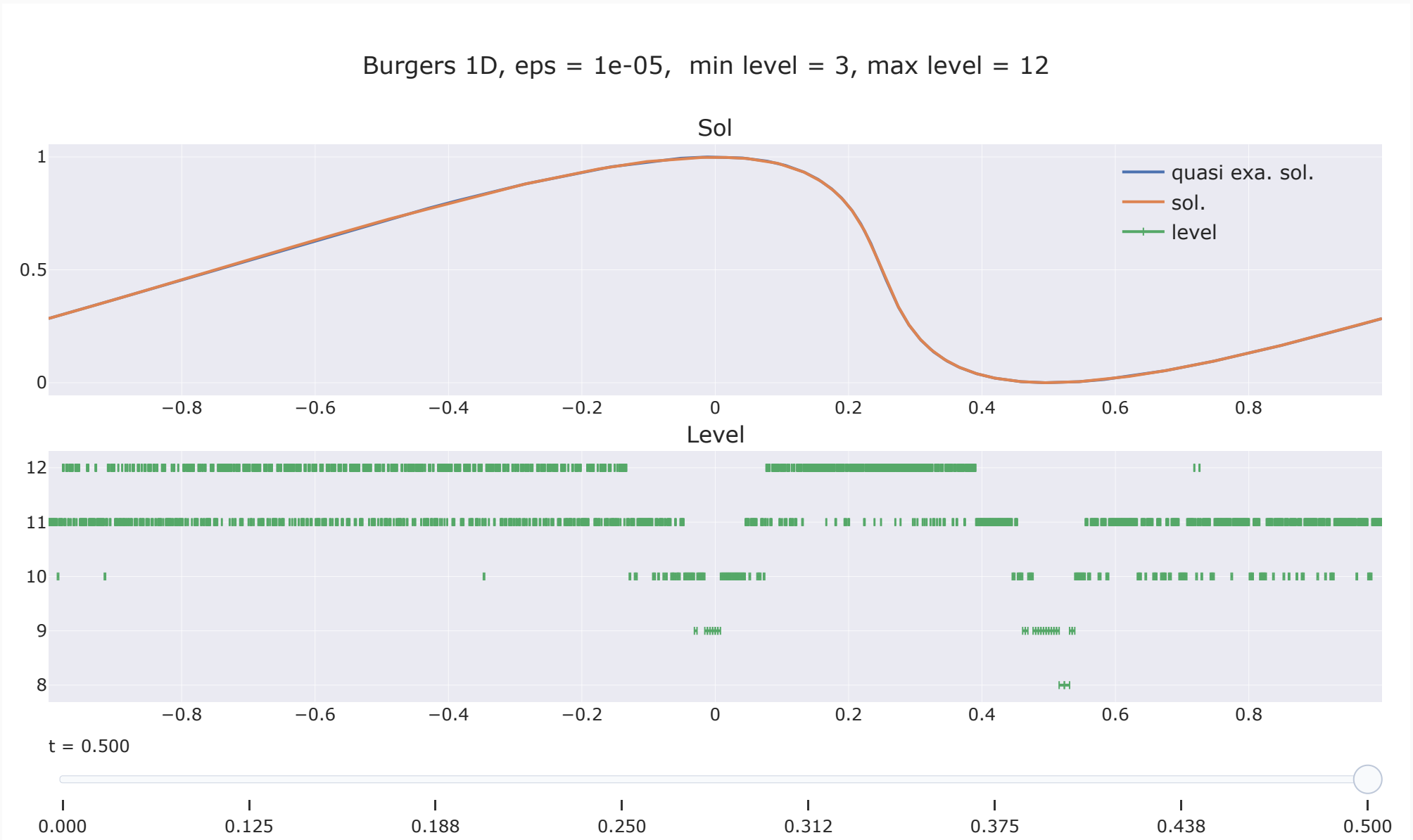
BURGERS RESULTS (MR+MLF SOL. AND ERROR FOR SCHEME ORDER 1)



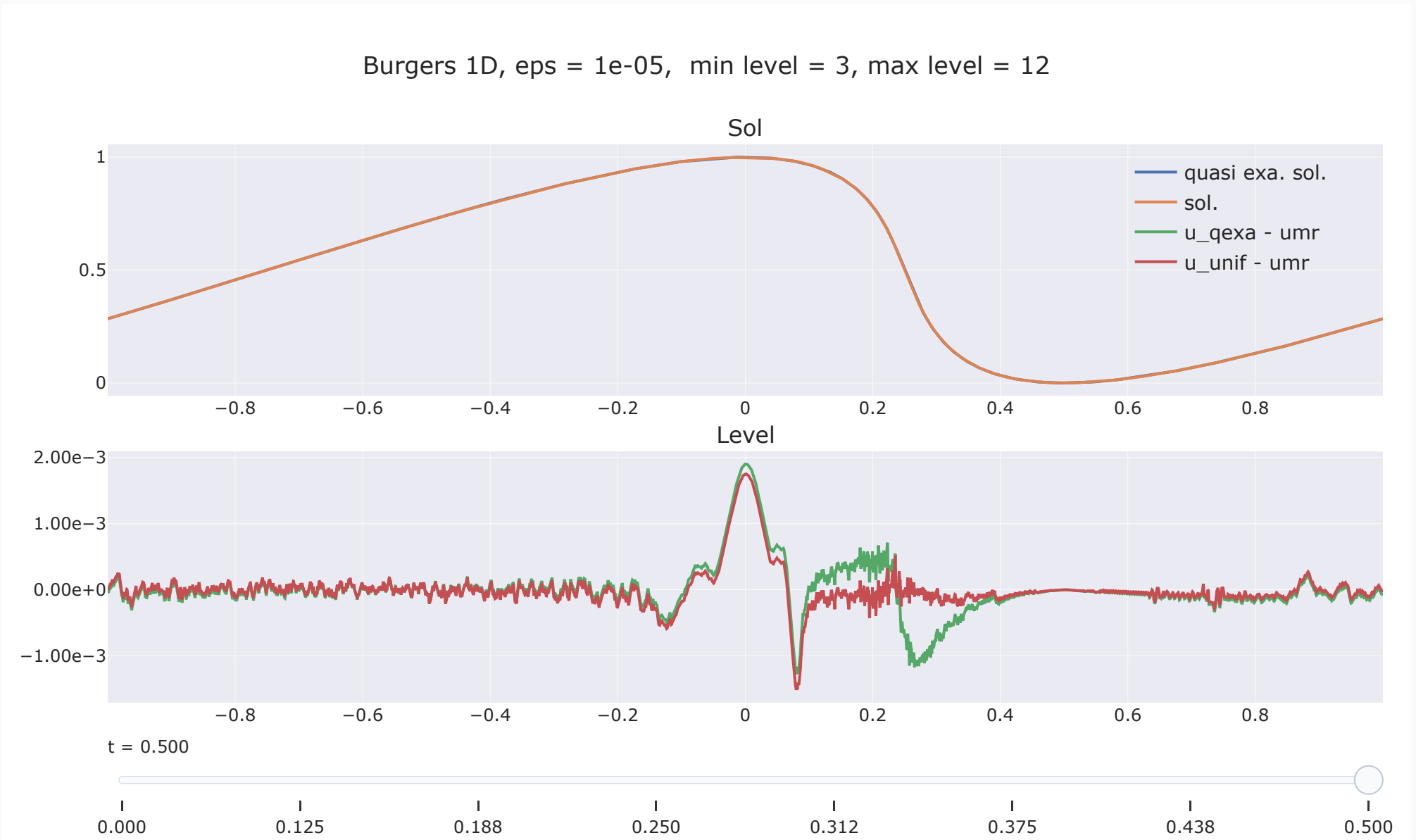
BURGERS RESULTS (ERROR FOR SCHEME ORDER 1)



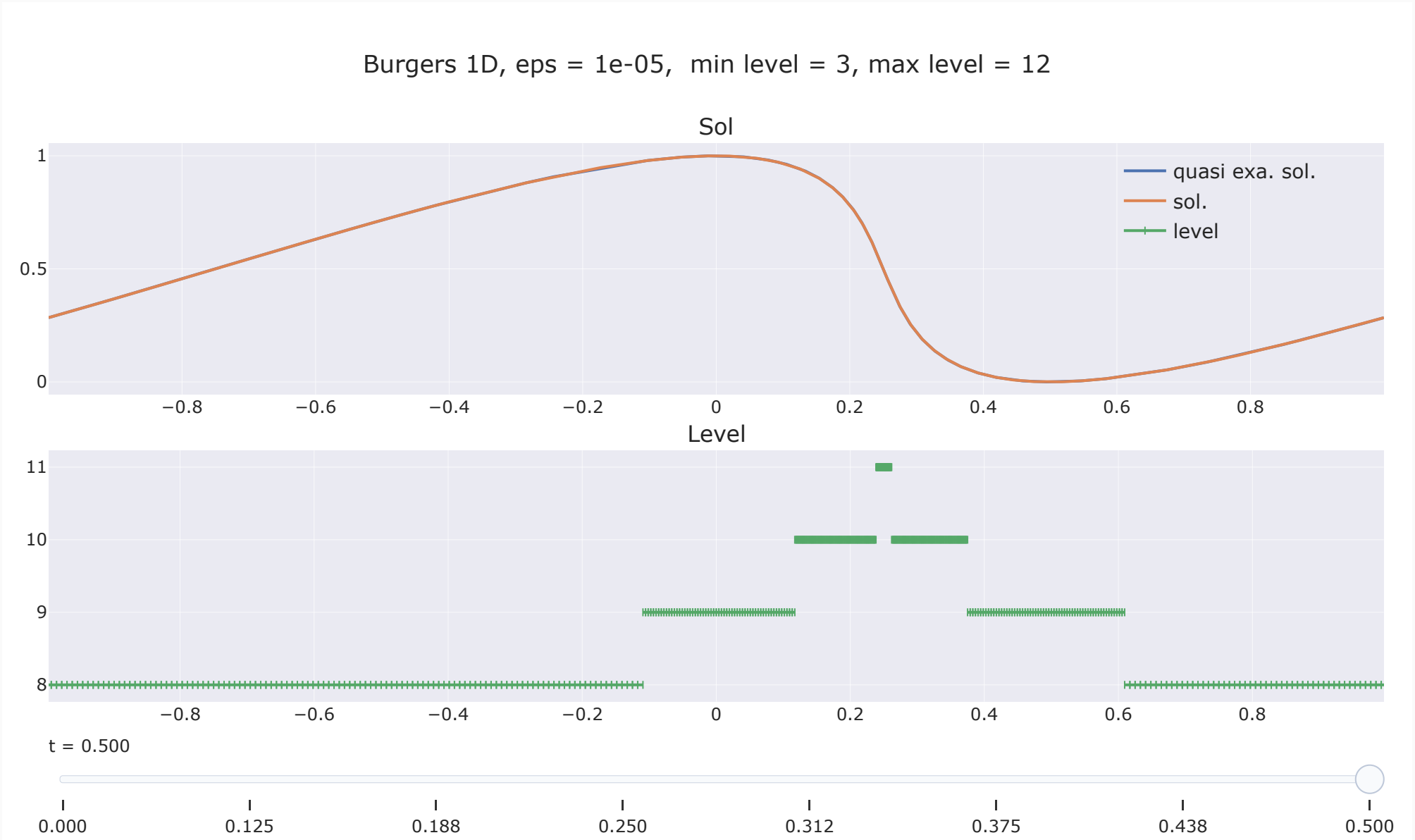
BURGERS RESULTS (MR SOLUTION AND LEVEL FOR SCHEME ORDER 1)



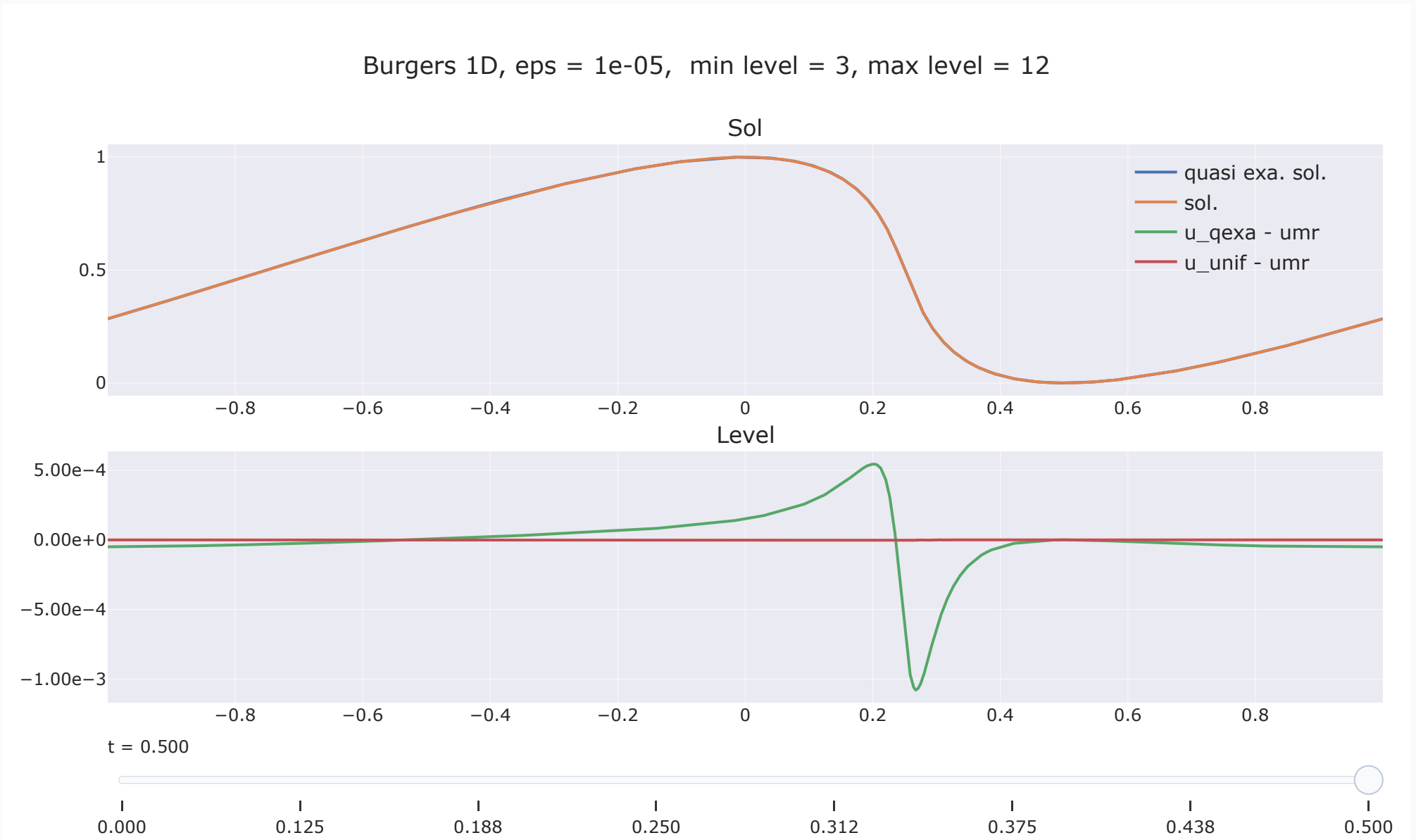
BURGERS RESULTS (MR SOLUTION AND ERROR FOR SCHEME ORDER 1)



BURGERS RESULTS (MR+MLF SOL. AND LEVEL FOR SCHEME ORDER 1)



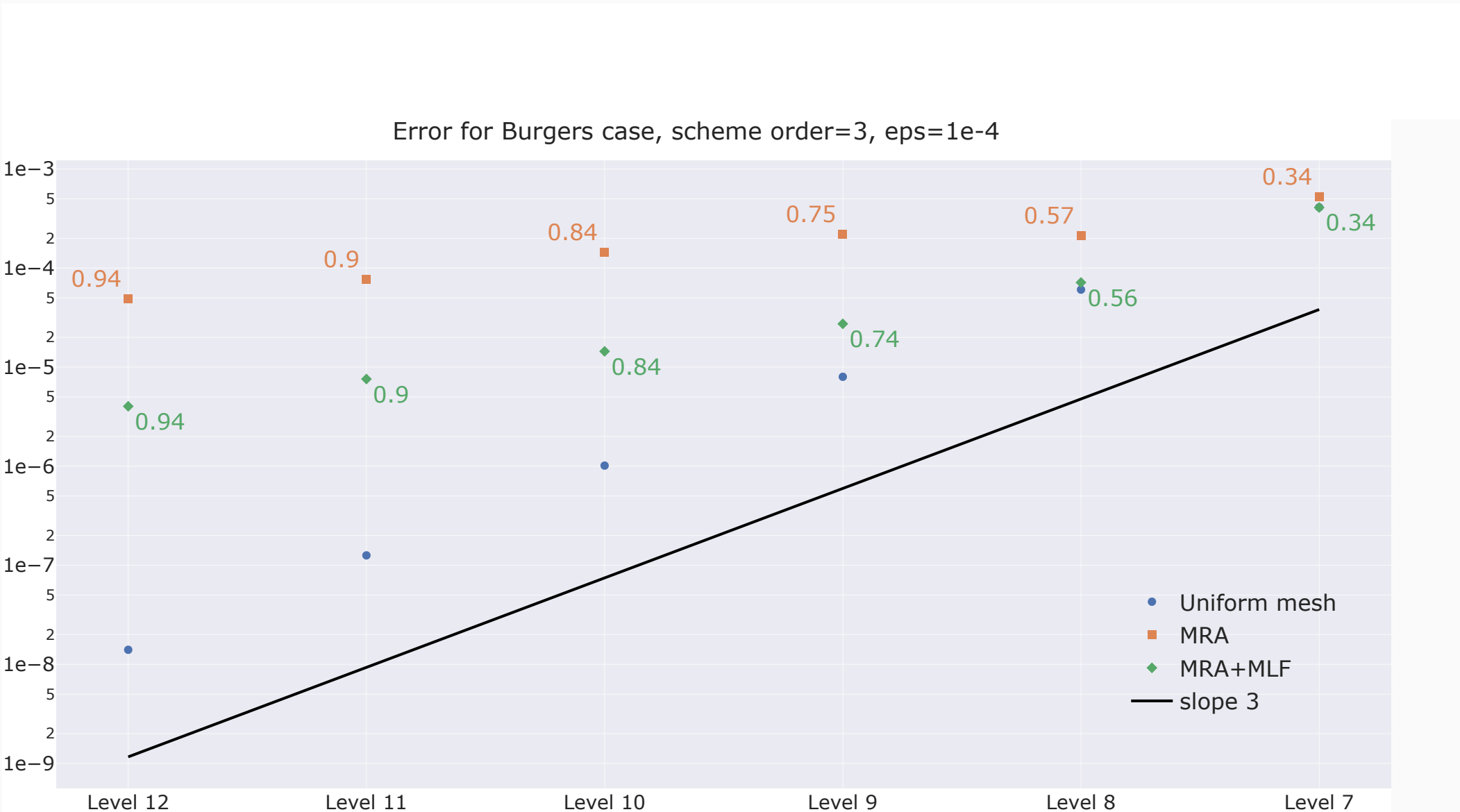
BURGERS RESULTS (MR+MLF SOL. AND ERROR FOR SCHEME ORDER 1)



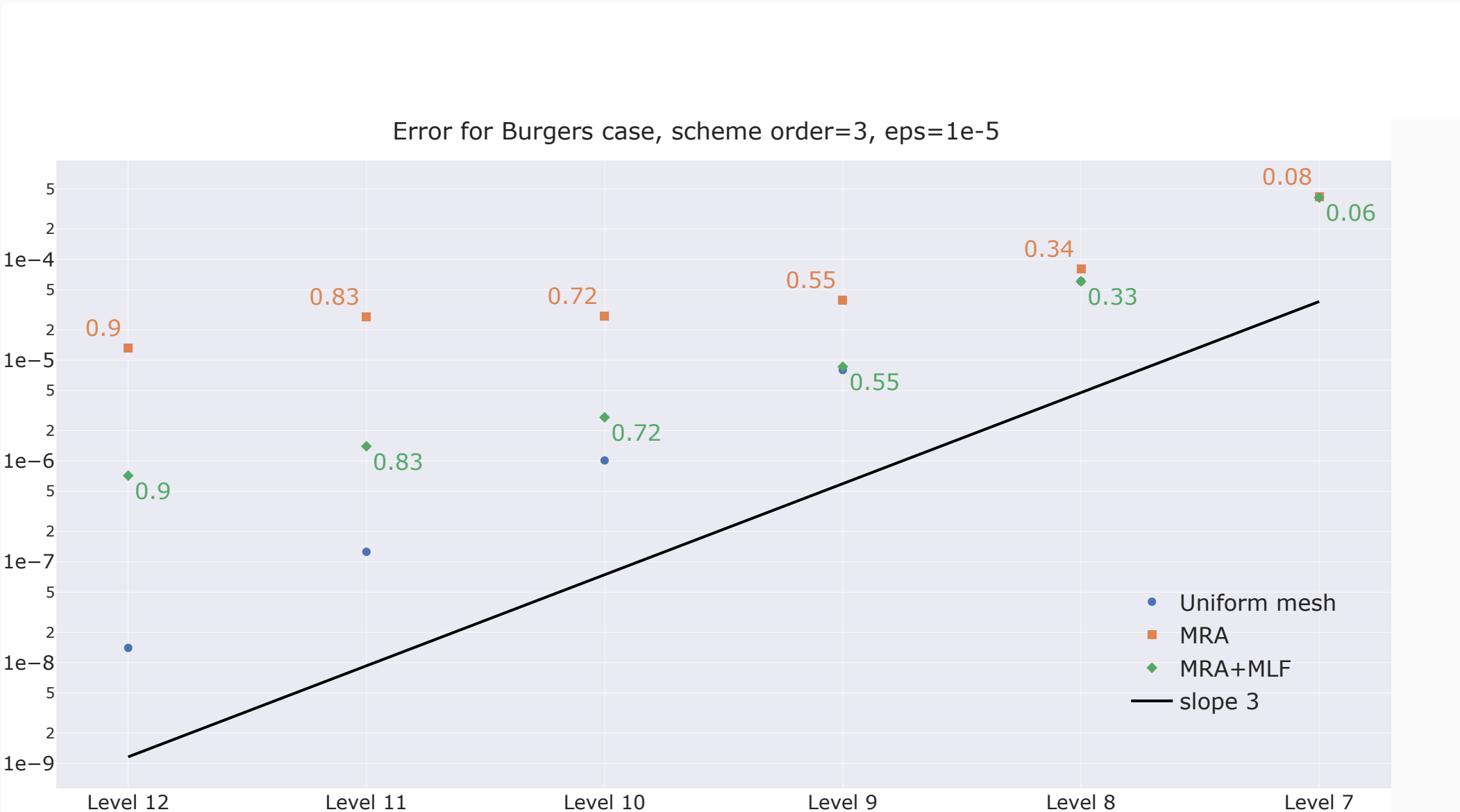
BURGERS RESULTS (ERROR FOR SCHEME ORDER 2)



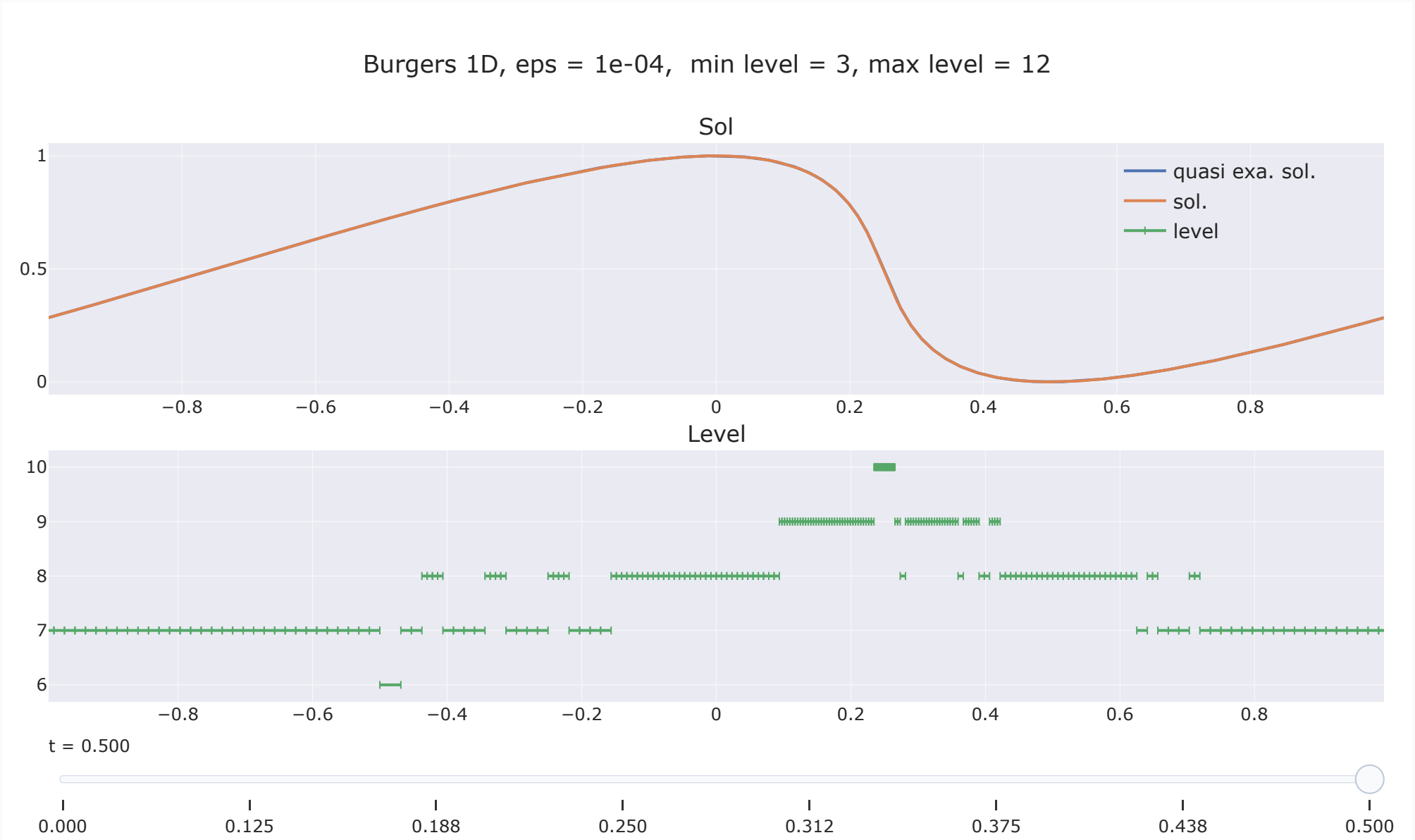
BURGERS RESULTS (ERROR FOR SCHEME ORDER 3)



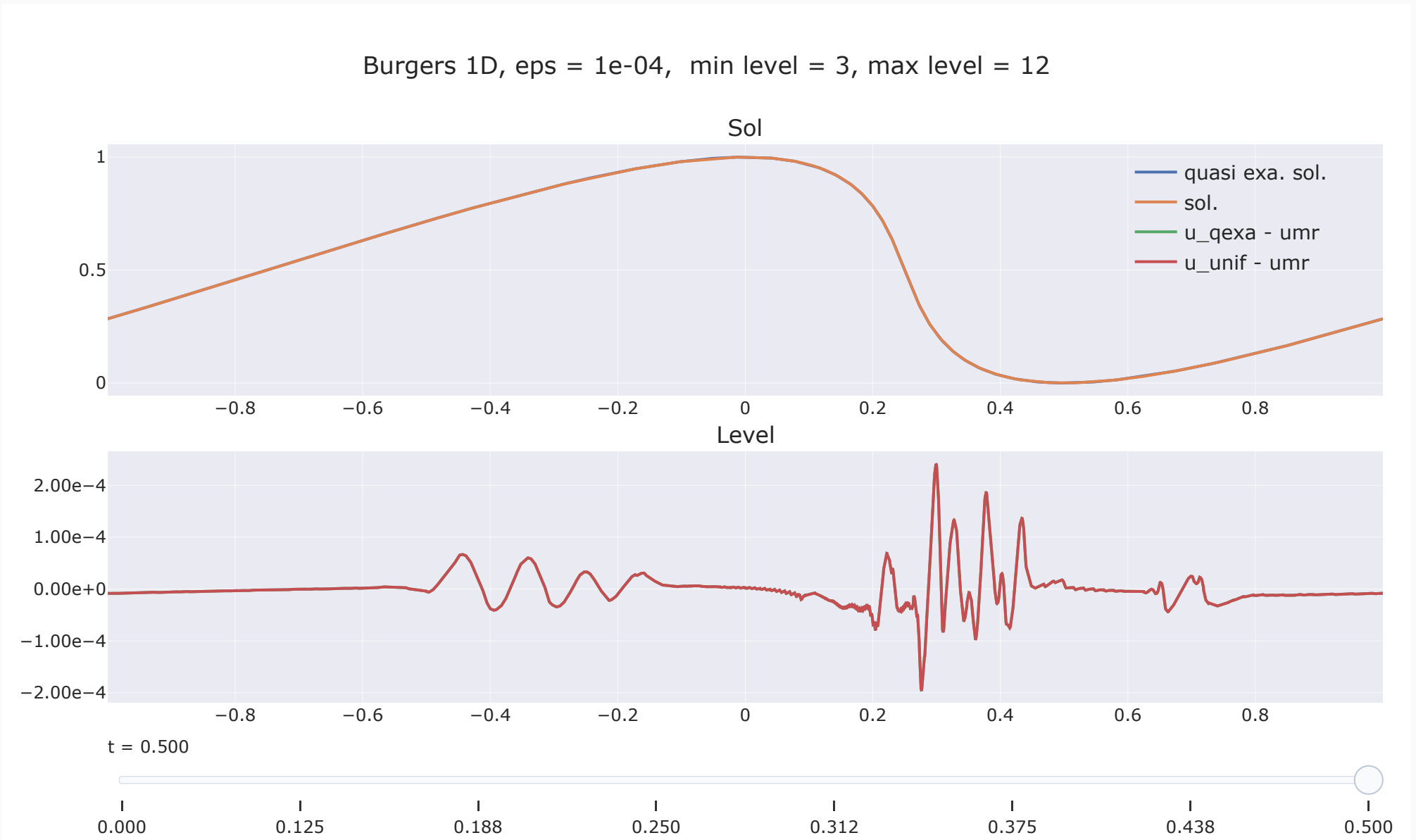
BURGERS RESULTS (ERROR FOR SCHEME ORDER 3)



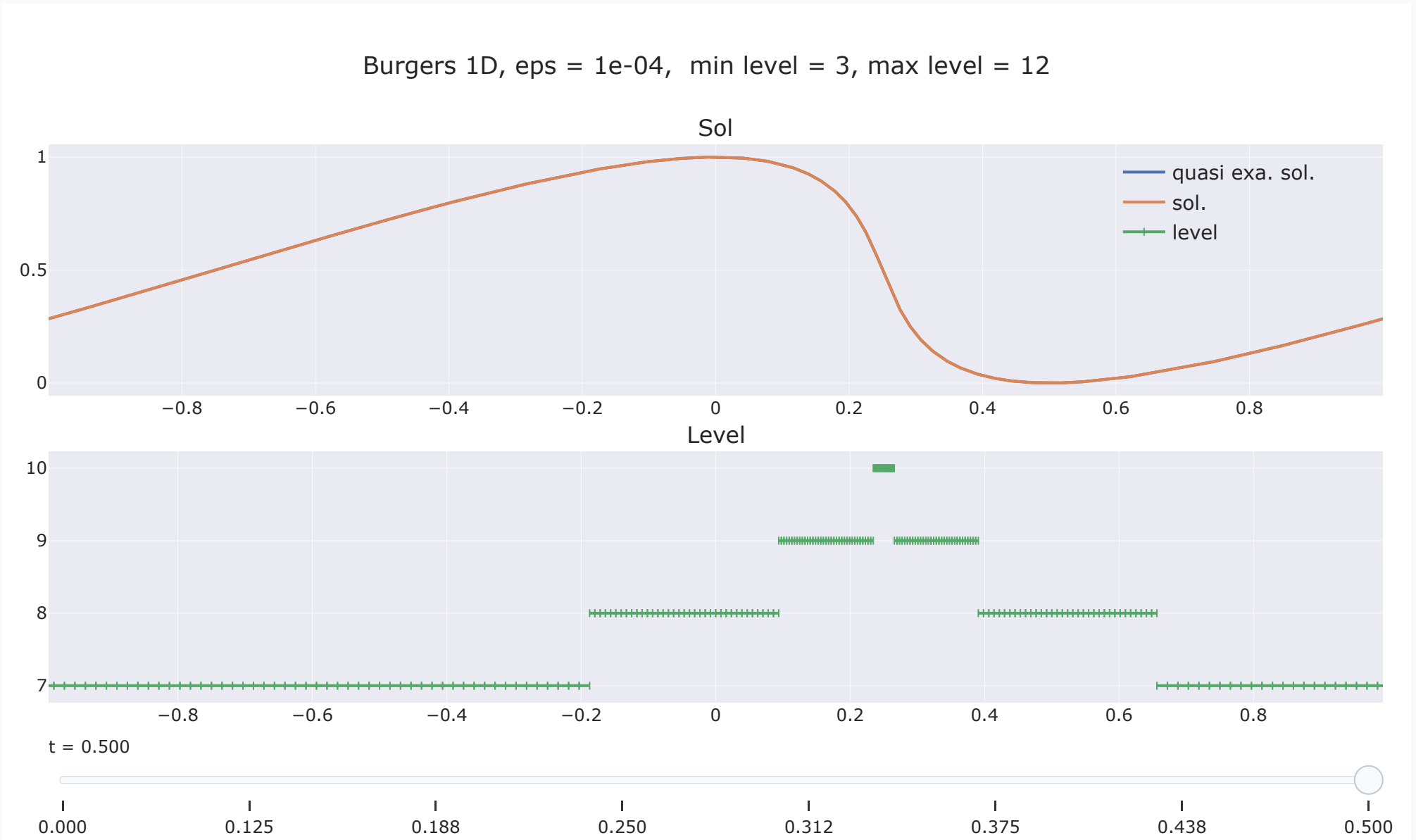
BURGERS RESULTS (MR SOLUTION AND LEVEL FOR SCHEME ORDER 3)



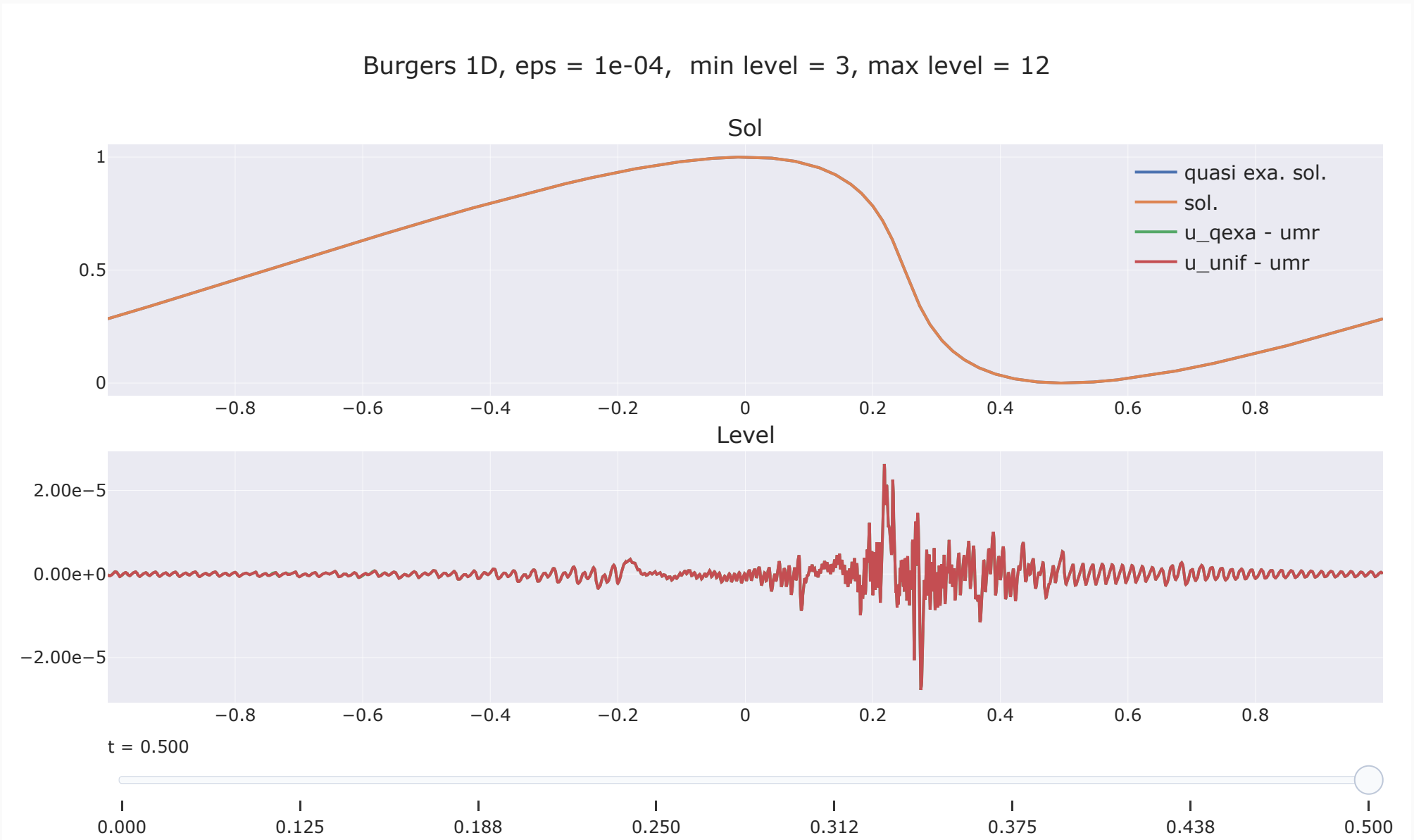
BURGERS RESULTS (MR SOLUTION AND ERROR FOR SCHEME ORDER 3)



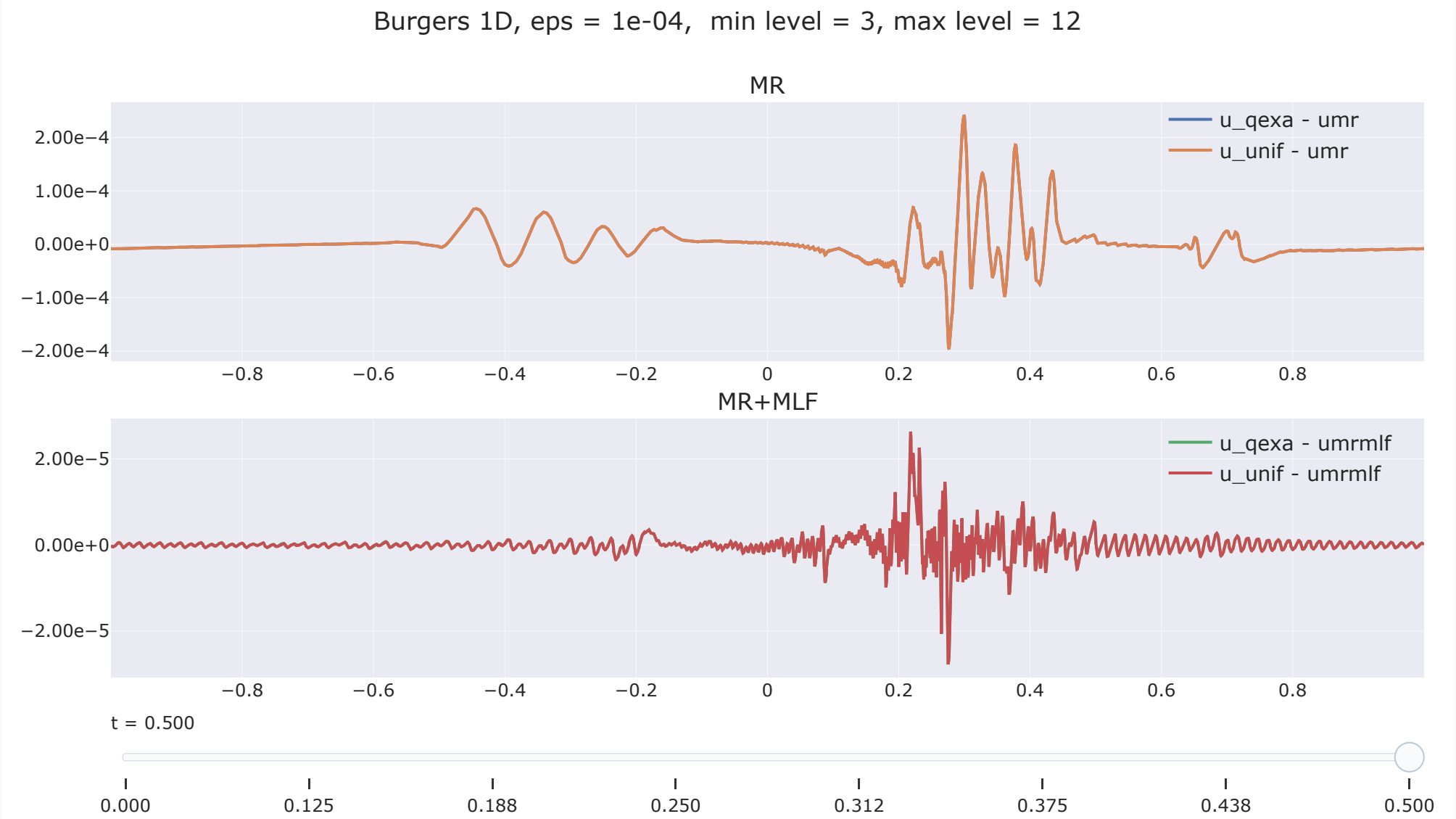
BURGERS RESULTS (MR+MLF SOL. AND LEVEL FOR SCHEME ORDER 3)



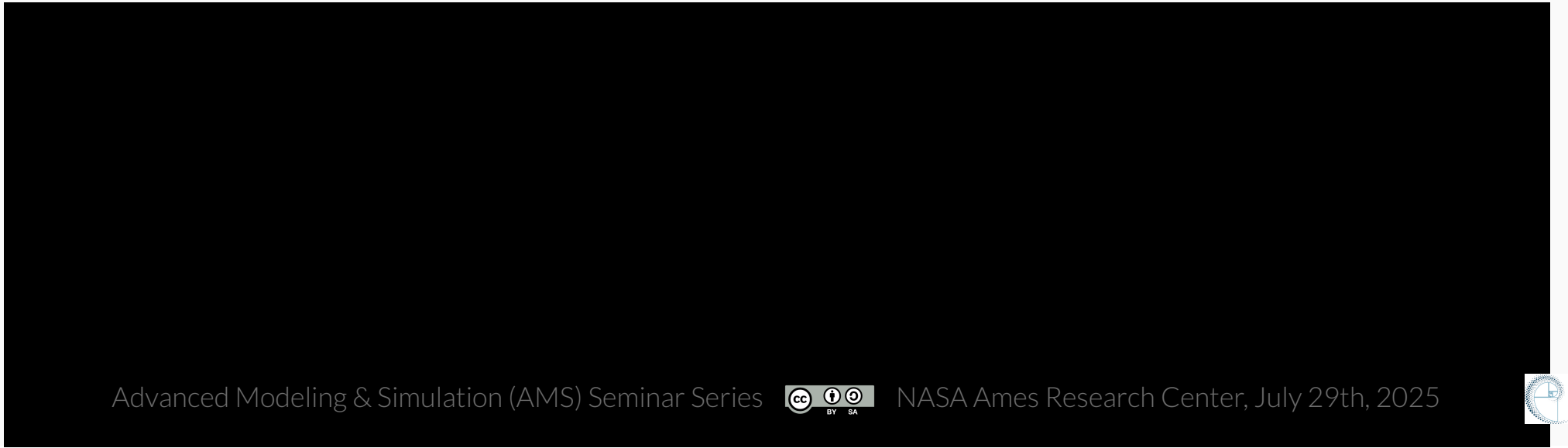
BURGERS RESULTS (MR+MLF SOL. AND ERROR FOR SCHEME ORDER 3)



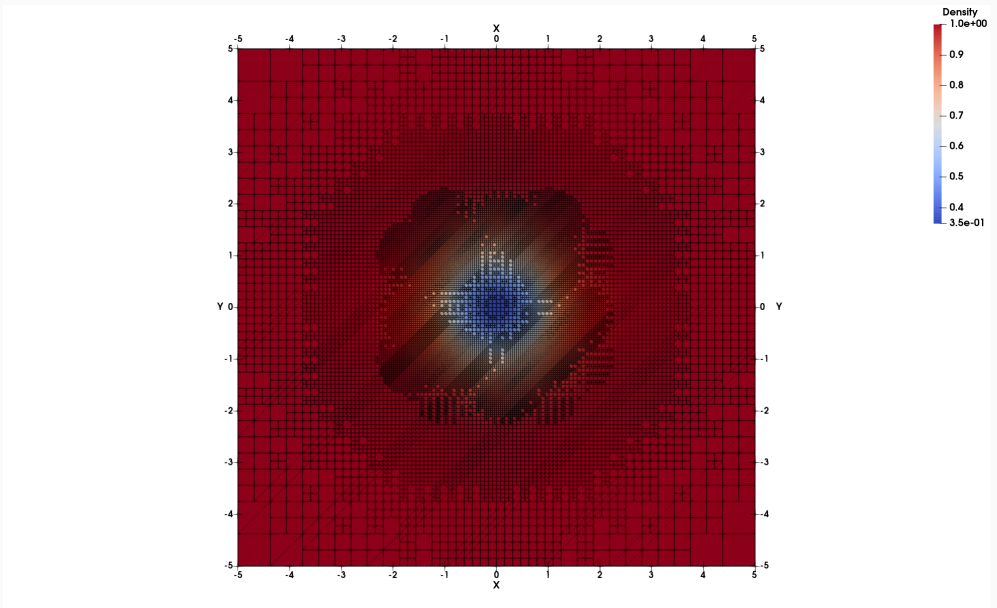
BURGERS RESULTS (MR VS MR+MLF ERROR FOR SCHEME ORDER 3)

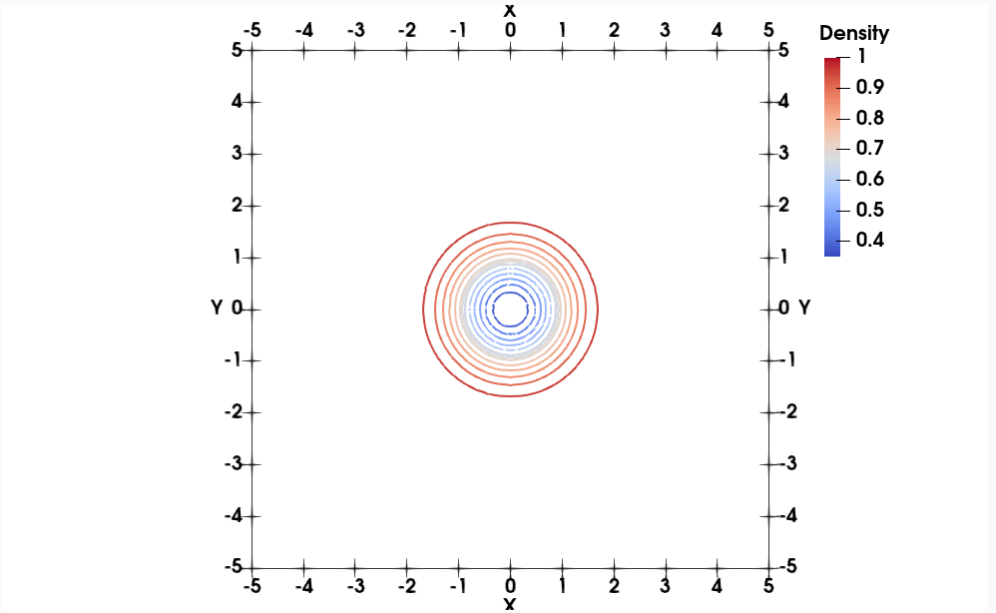


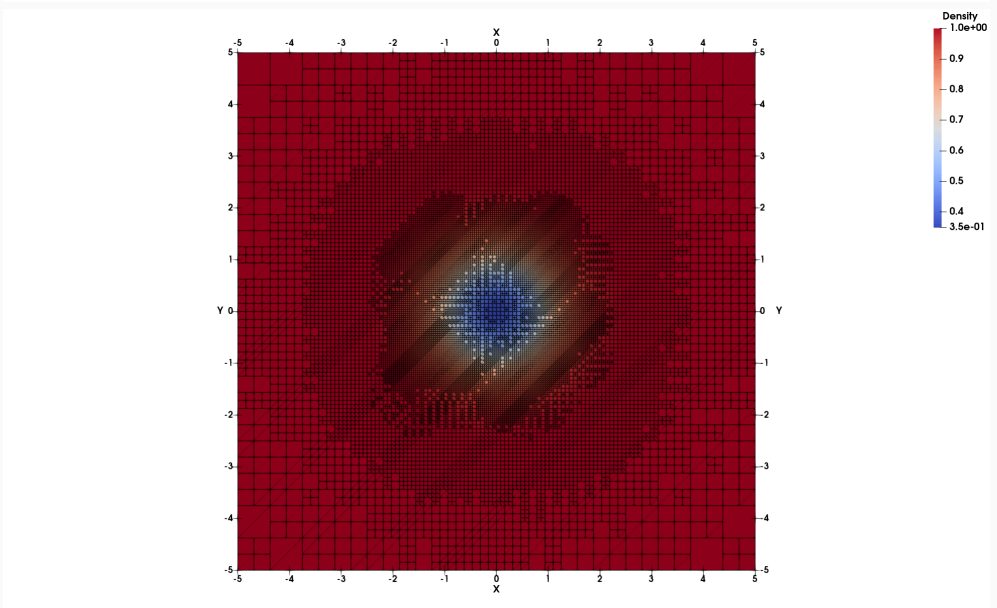
BURGERS 2D RESULTS (MR+MLF SOLUTION ORDER 1)

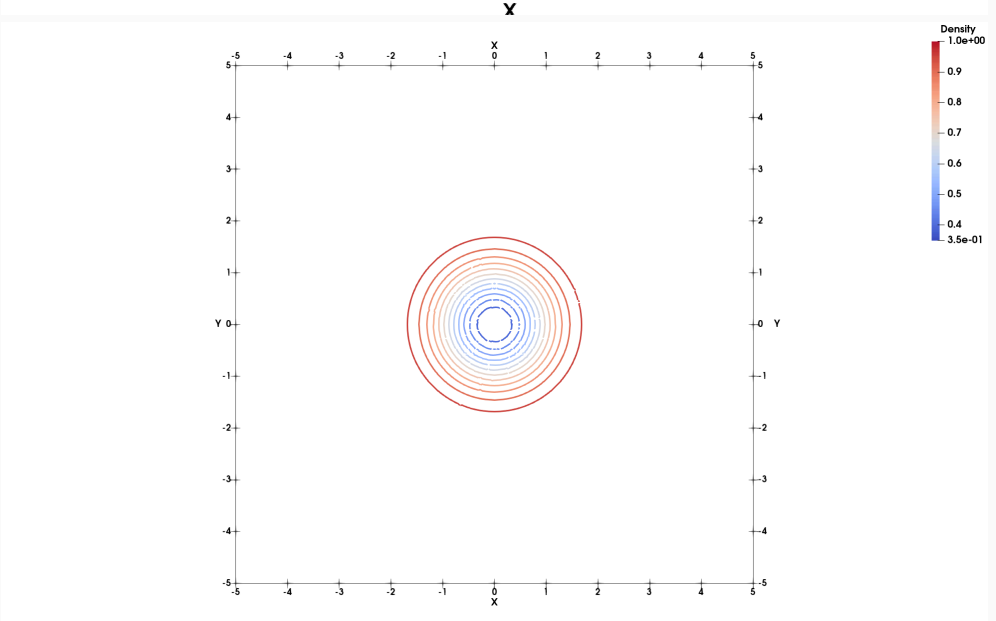


EULER 2D RESULTS (MR+MLF SOLUTION ORDER 3 (OSMP SCHEME))

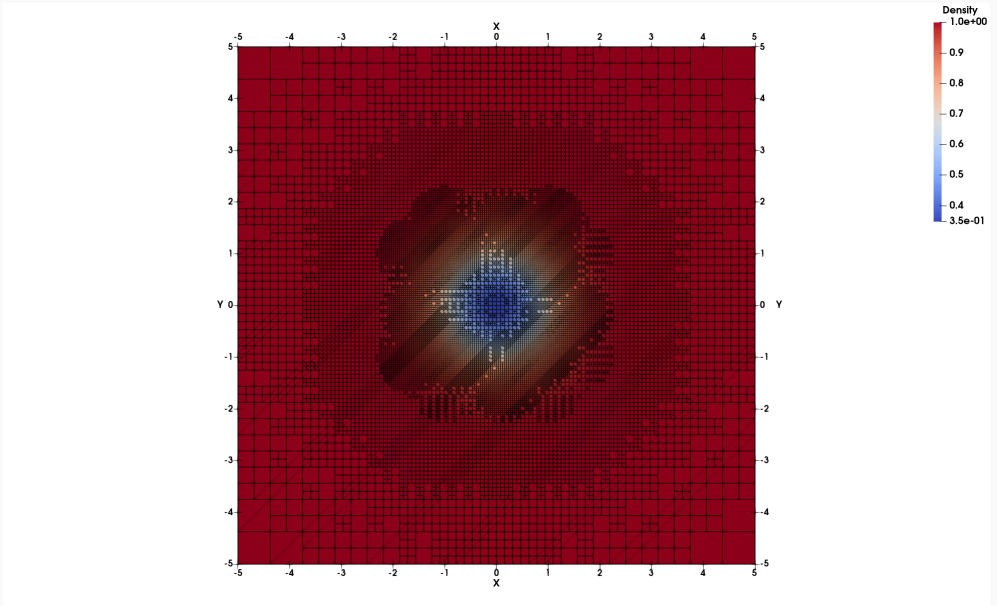


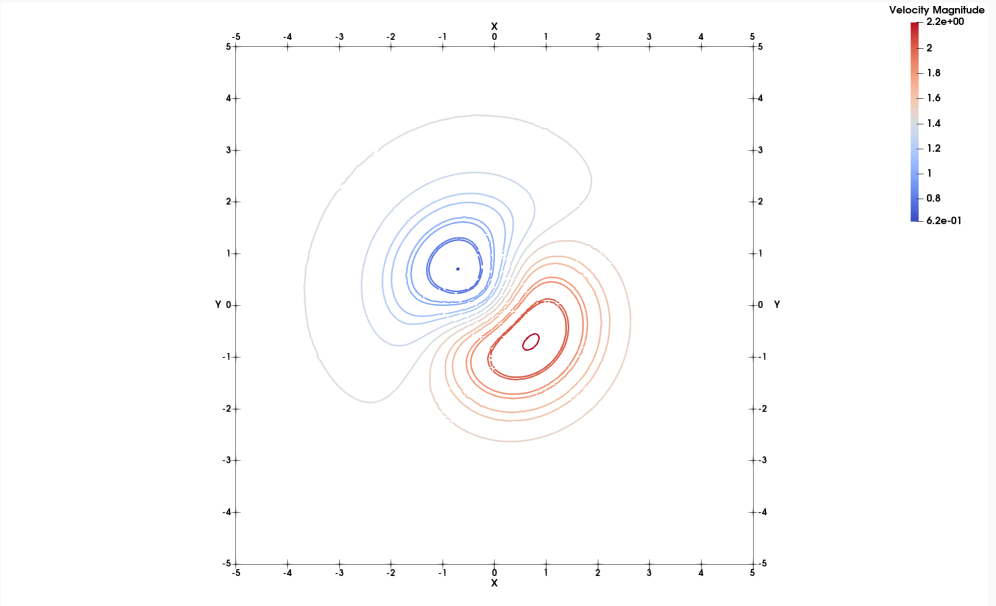


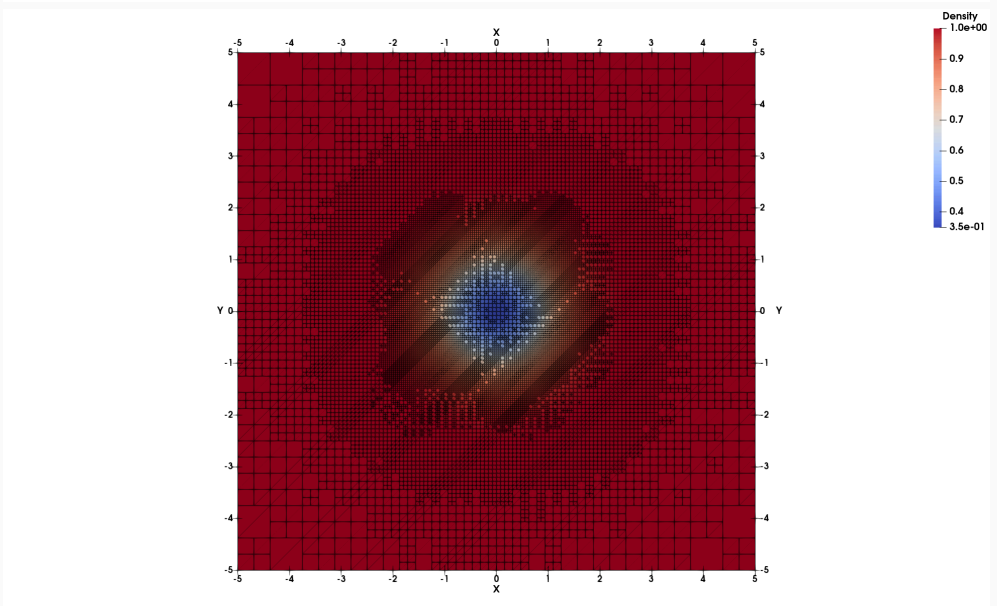


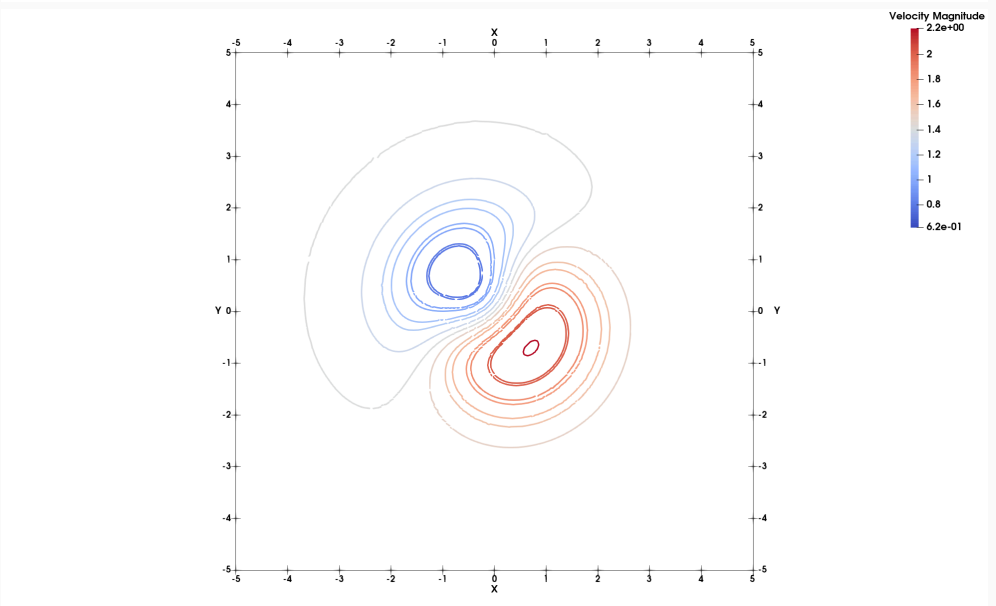


EULER 2D RESULTS (MR+MLF SOLUTION ORDER 3 (OSMP SCHEME))



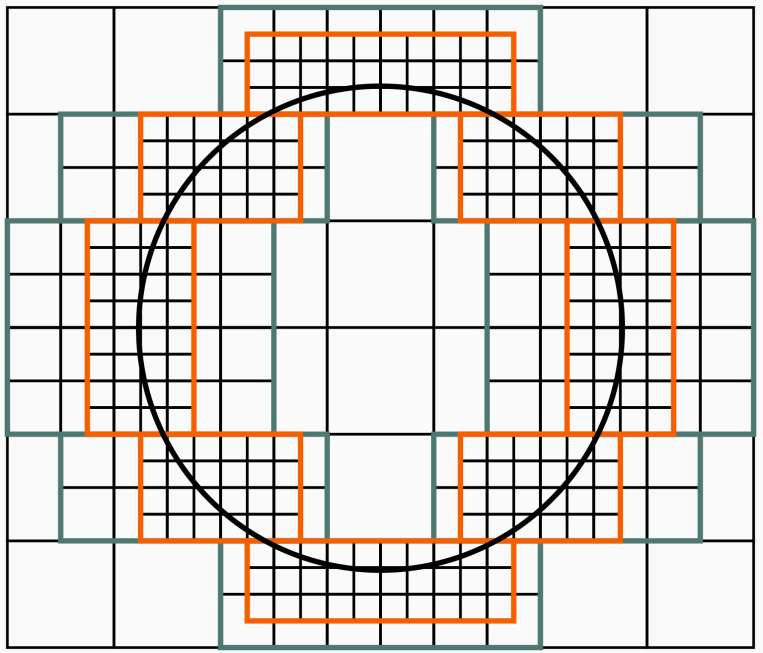






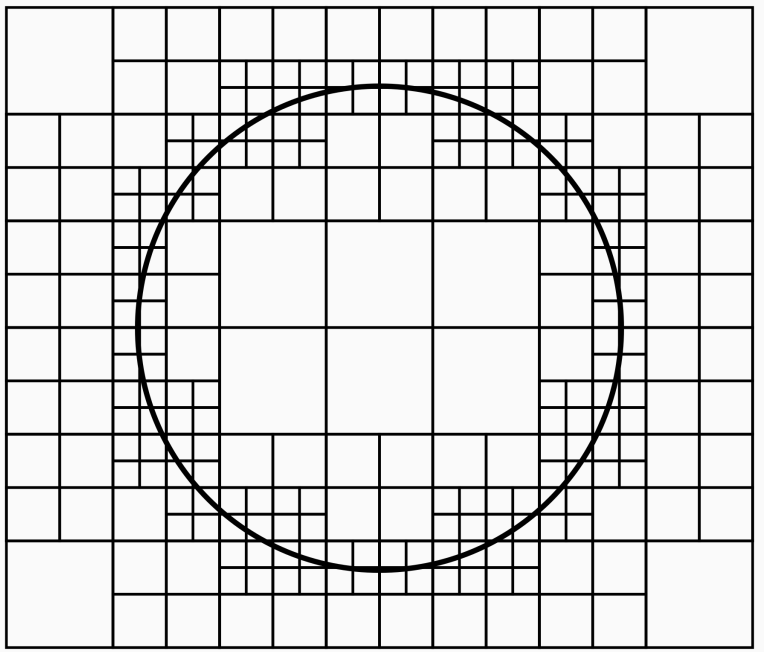
ADAPTIVE MESH REFINEMENT SOFTWARE

MESH ADAPTATION



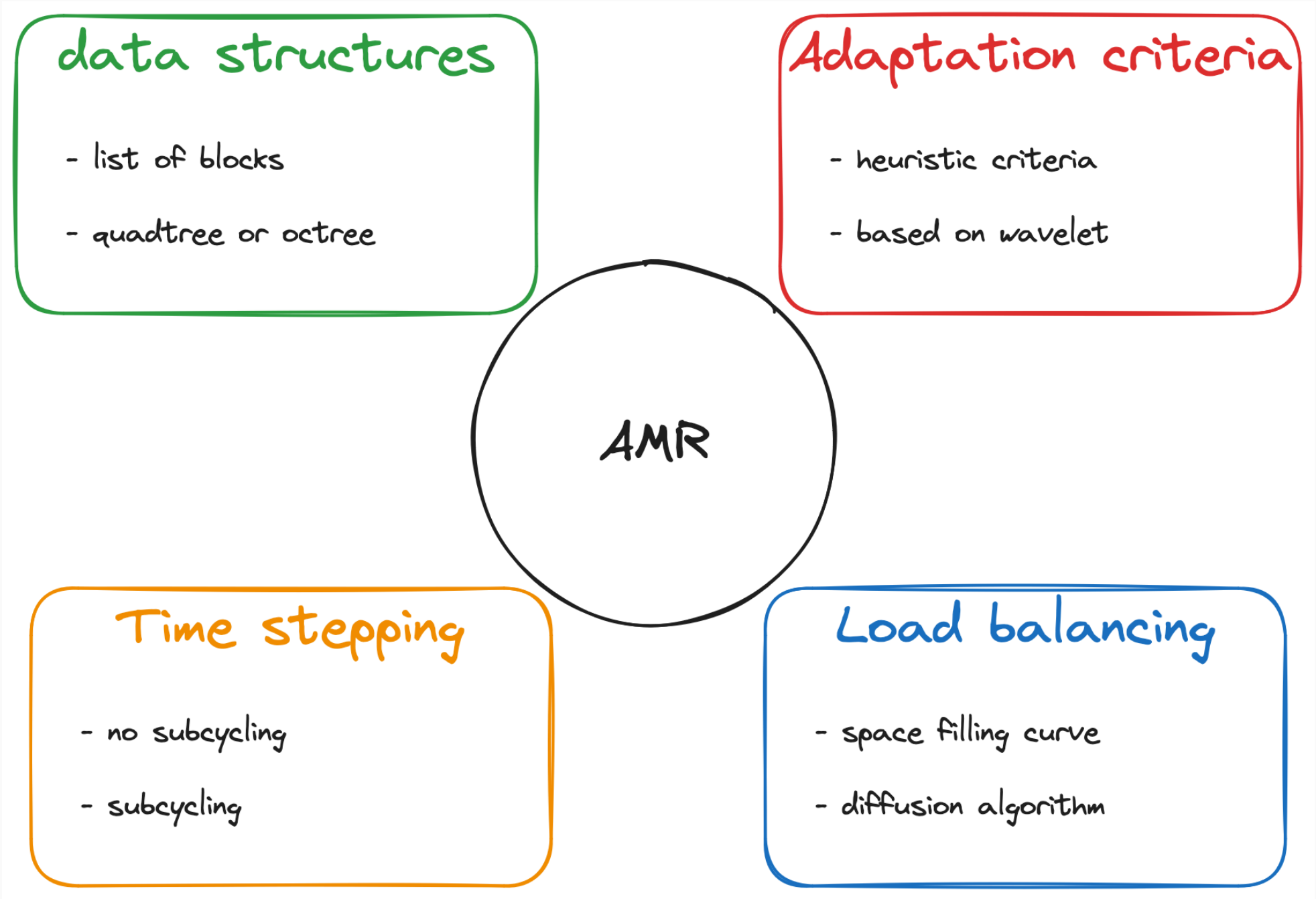
patch-based

AMR



cell-based

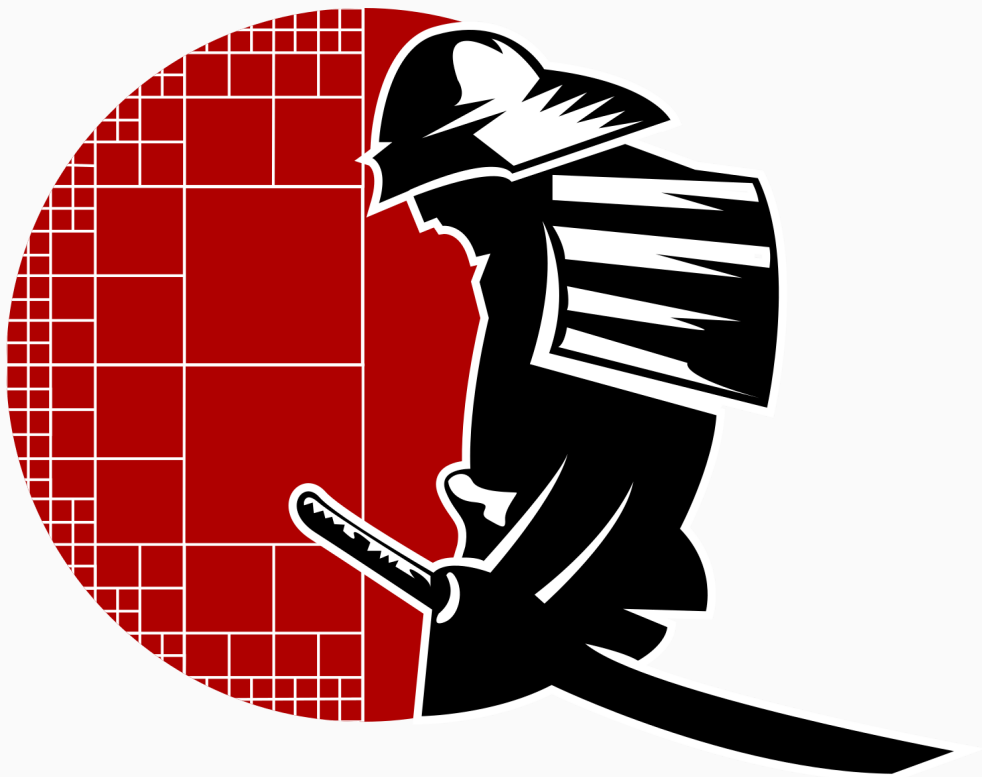
AMR and multiresolution


data structures
- list of blocks
- quadtree or octree
Adaptation criteria
- heuristic criteria
- based on wavelet
AMR
Time stepping
- no subcycling
- subcycling
Load balancing
- space filling curve
- diffusion algorithm

OPEN SOURCE SOFTWARE

Name	Data structure	Adaptation criteria	Time scheme	Load balancing
AMReX	block	heuristic	global/local	SFC
Dendro	tree	wavelet	global	SFC
Dyablo	tree	heuristic	global	SFC
Peano	tree	-	-	SFC
P4est	tree	-	-	SFC
samurai	interval	heuristic/wavelet	RK/splitting/IMEX time-space/code coupling	SFC/diffusion algorithm

samurai: create a unified framework for testing a whole range of mesh adaptation methods with the latest generation of numerical schemes.



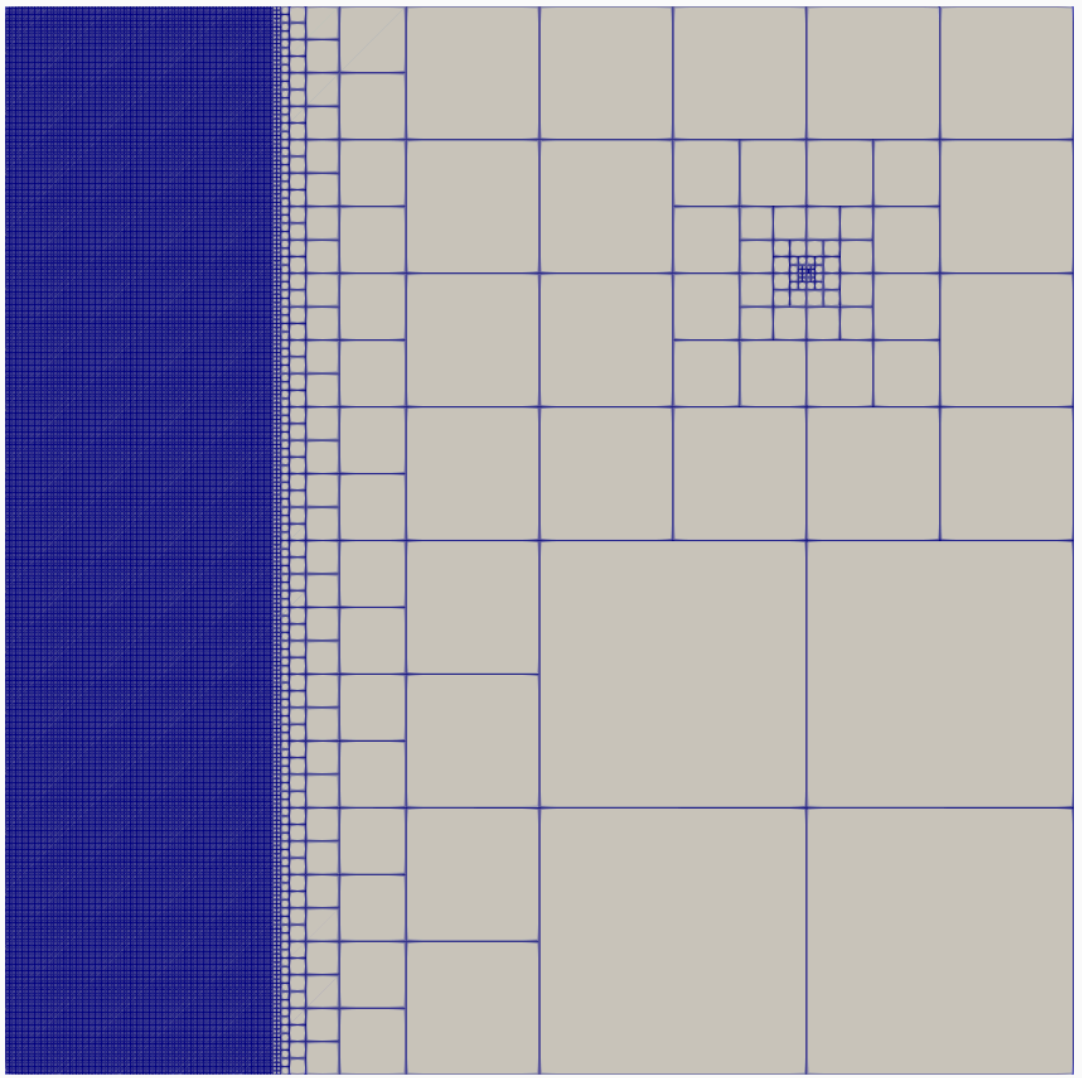
SAMURAI

CC BY SA

DESIGN PRINCIPLES

- Compress the mesh according to the level-wise spatial connectivity along each Cartesian axis.
- Achieve fast look-up for a cell into the structure, especially for parents and neighbors.
- Maximize the memory contiguity of the stored data to allow for caching and vectorization.
- Facilitate inter-level operations which are common in many numerical techniques.

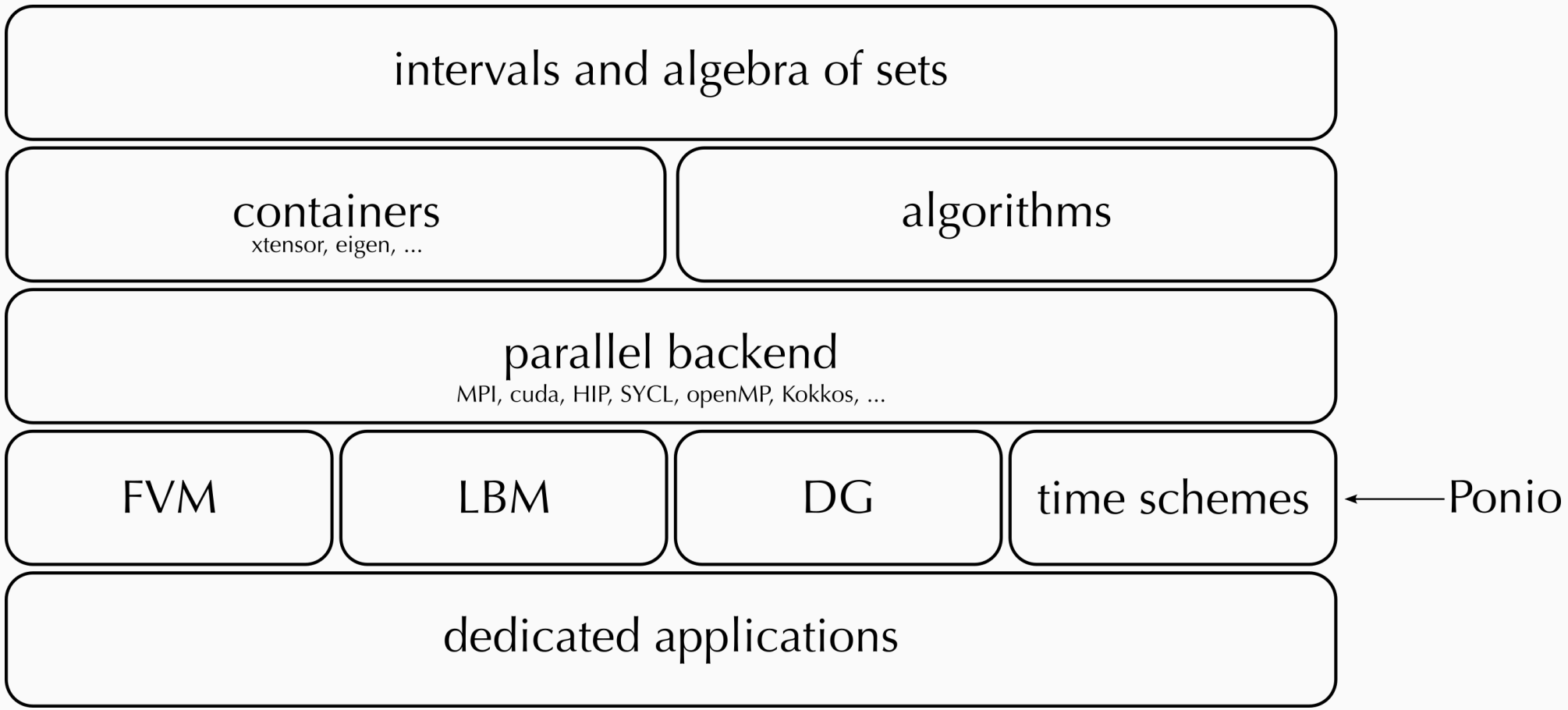
COMPRESSION RATES



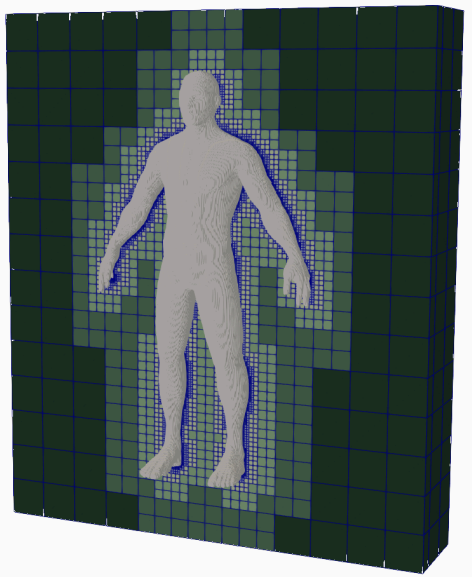
COMPRESSION RATES

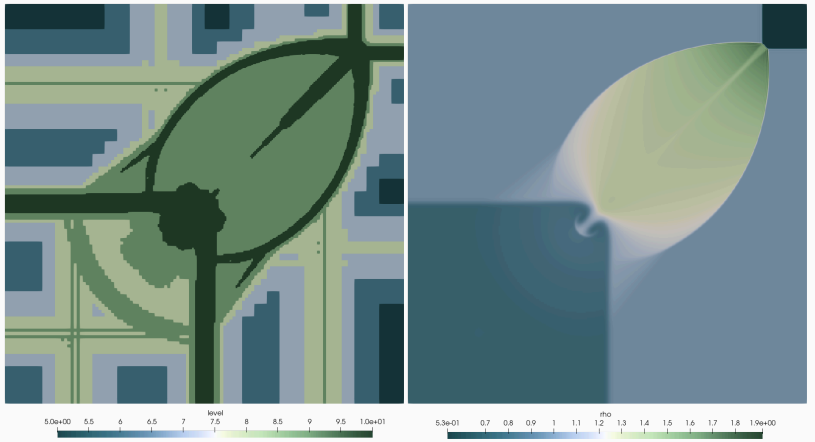
Level	Num. of cells	p4est	samurai (leaves)	samurai (all)	ratio
9	66379	2.57 Mb	33.68 Kb	121 Kb	21.24
10	263767	10.25 Mb	66.64 Kb	236.8 Kb	43.28
11	1051747	40.96 Mb	132.36 Kb	467.24 Kb	87.66
12	4200559	163.75 Mb	263.6 Kb	927 Kb	176.64
13	16789627	654.86 Mb	525.9 Kb	1.85 Mb	353.98
14	67133575	2.61 Gb	1.05 Mb	3.68 Mb	709.24

ROADMAP

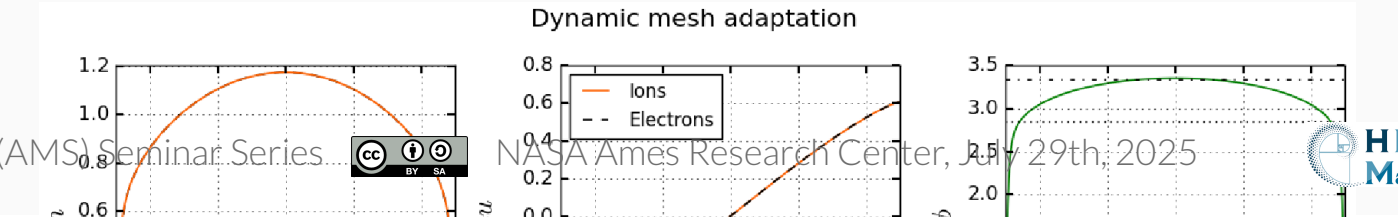

intervals and algebra of sets
containers
xtensor, eigen, ...
algorithms
parallel backend
MPI, cuda, HIP, SYCL, openMP, Kokkos, ...
FVM
LBM
DG
time schemes
Ponio
dedicated applications






level
rho




Dynamic mesh adaptation
Ions
Electrons



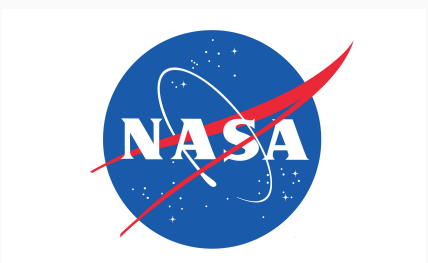
SCIENTIFIC COLLABORATIONS

- Lattice Boltzmann methods and multiresolution - **Thomas Bellotti** (*EM2C/CNRS/CS*) and **Benjamin Graille** (*LMO/Université Paris-Saclay*)
- Plasma discharges and electric propulsion - **Alejandro Alvarez-Laguna** (*LPP/École polytechnique*) and **Louis Reboul** (*ONERA*) **Teddy Pichard** and **Zoubaïr Tazakkati** (*CMAP/École polytechnique*)
- DNS of lithium-ion batteries based on high-resolution 3D images of porous electrode microstructures - **Ali Asad** (*TotalEnergies*) and **Laurent François** (*ONERA*)
- Sharp interface method for low Mach two-phase flows - **Nicolas Grenier** (*LISN/Université Paris-Saclay*) and **Christian Tenaud** (*EM2C/CNRS/CS*)
- Low-Mach reactive flows - **Christian Tenaud** (*EM2C/CNRS/CS*)
- Interfacial flow simulation relying on diffuse interface / two-scale modeling - **Giuseppe Orlando**, **Ward Haegeman** (*CMAP/Ecole polytechnique*), **Samuel Kokh** (*CEA/MdIS*), **Joël Dupays**, **Clément Le Touze** (*ONERA*), **Marica Pelanti** (*ENSTA/IP Paris*), **Khaled Saleh** (*Aix-Marseille Université*), **Jean-Marc Hérard** (*EDF*)
- High-fidelity simulation of the Hydrogen risk - **Luc Lecointre**, **Pierre-Alexandre Masset**, **Etienne Studer** (*CEA*), **Sergey Koudriakov** (*CEA*) and **Christian Tenaud** (*EM2C/CNRS/CS*)

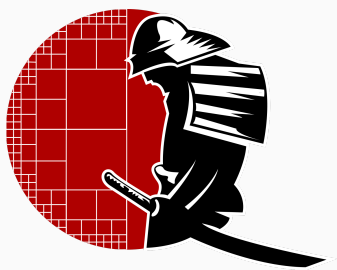
FIVE PROJECTS - NASA MODELLING SUMMER VISIT 2025

Five groups from France involving 15 students / post-doctoral fellows / research software engineers / faculties

- Mathematical modeling and simulation of non-equilibrium plasmas for the prediction of electric propulsion and plasma discharges including sheath
- Code coupling : multistep coupling algorithm (Tuesday's seminar)
- A two-phase flow model with scale separation for the separated-to-disperse phase transition in high-velocity regimes
- A dual splitting/ImEx strategy for multicomponent reacting flows with space/time adaptation and error control
- Simulation of hybrid dense-rarefied flows relying on relaxation schemes and a hierarchy of moment methods





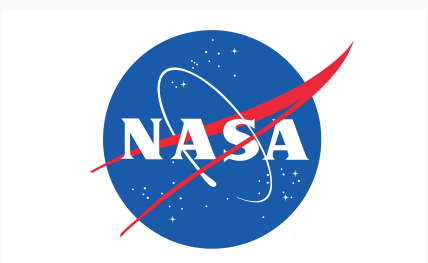




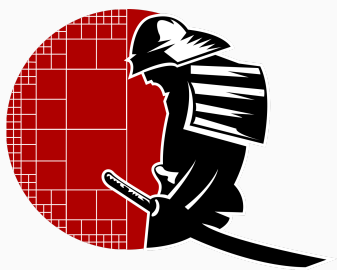
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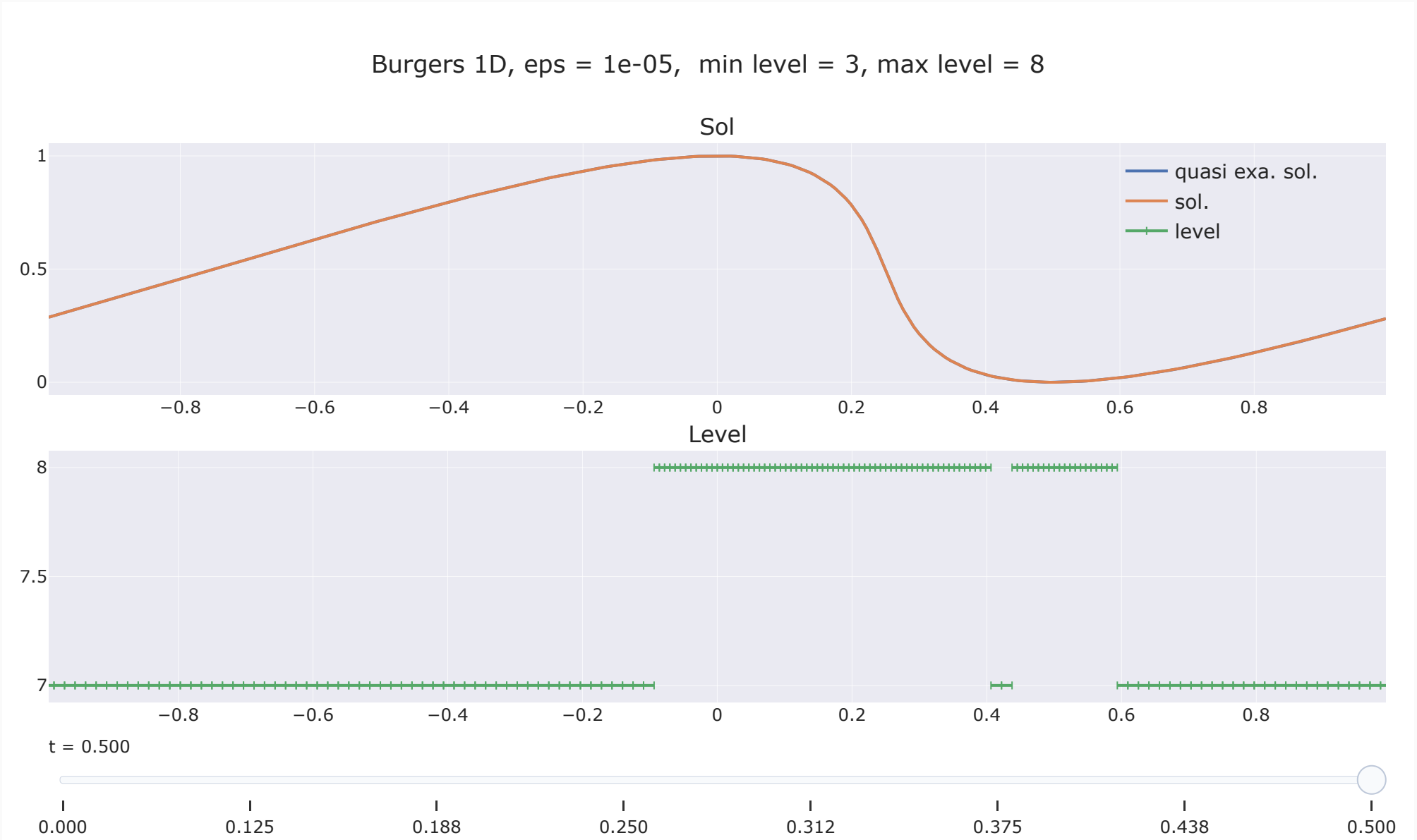




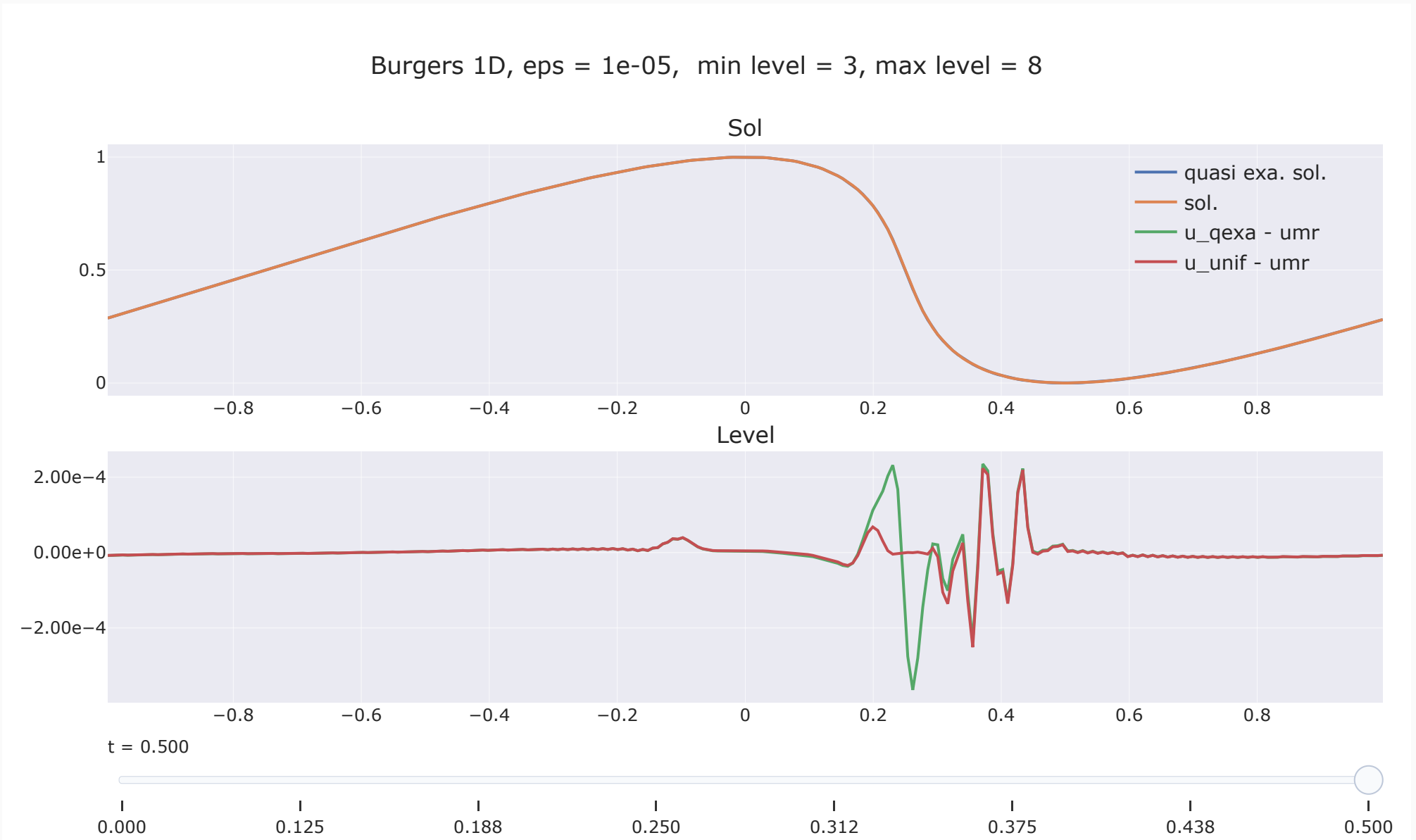
THANK YOU FOR YOUR ATTENTION

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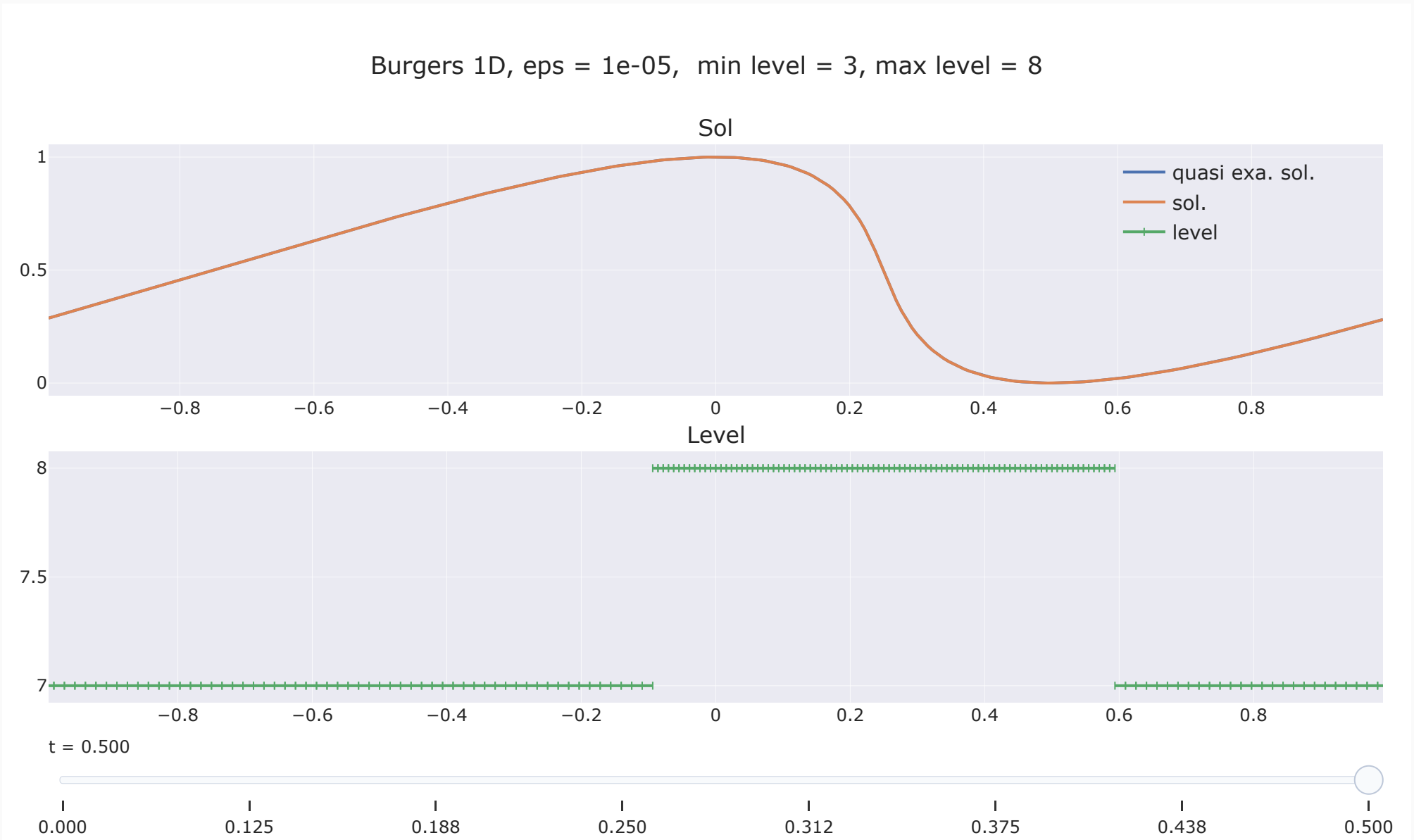
BURGERS RESULTS (MR SOLUTION AND LEVEL FOR SCHEME ORDER 3)



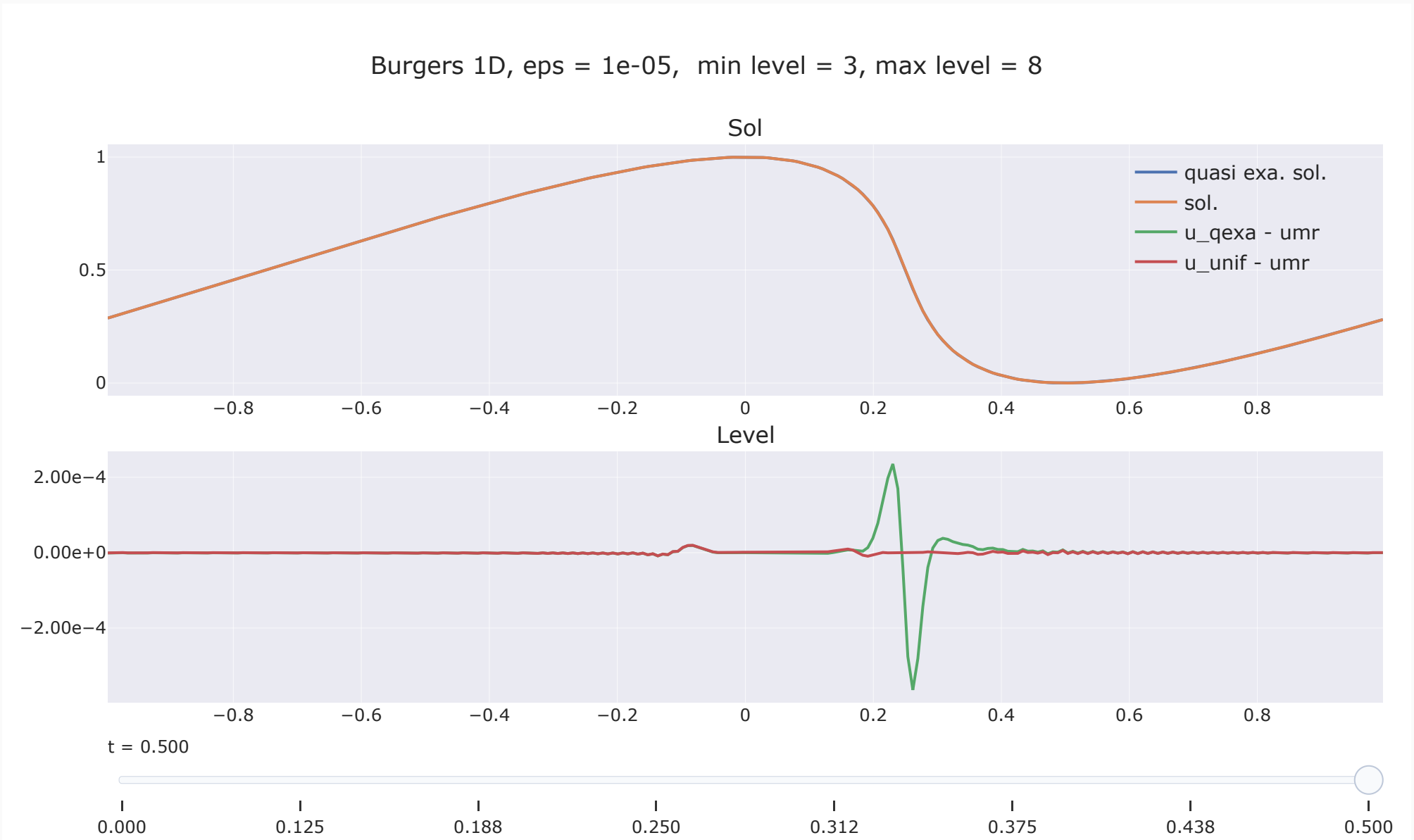
BURGERS RESULTS (MR SOLUTION AND ERROR FOR SCHEME ORDER 3)



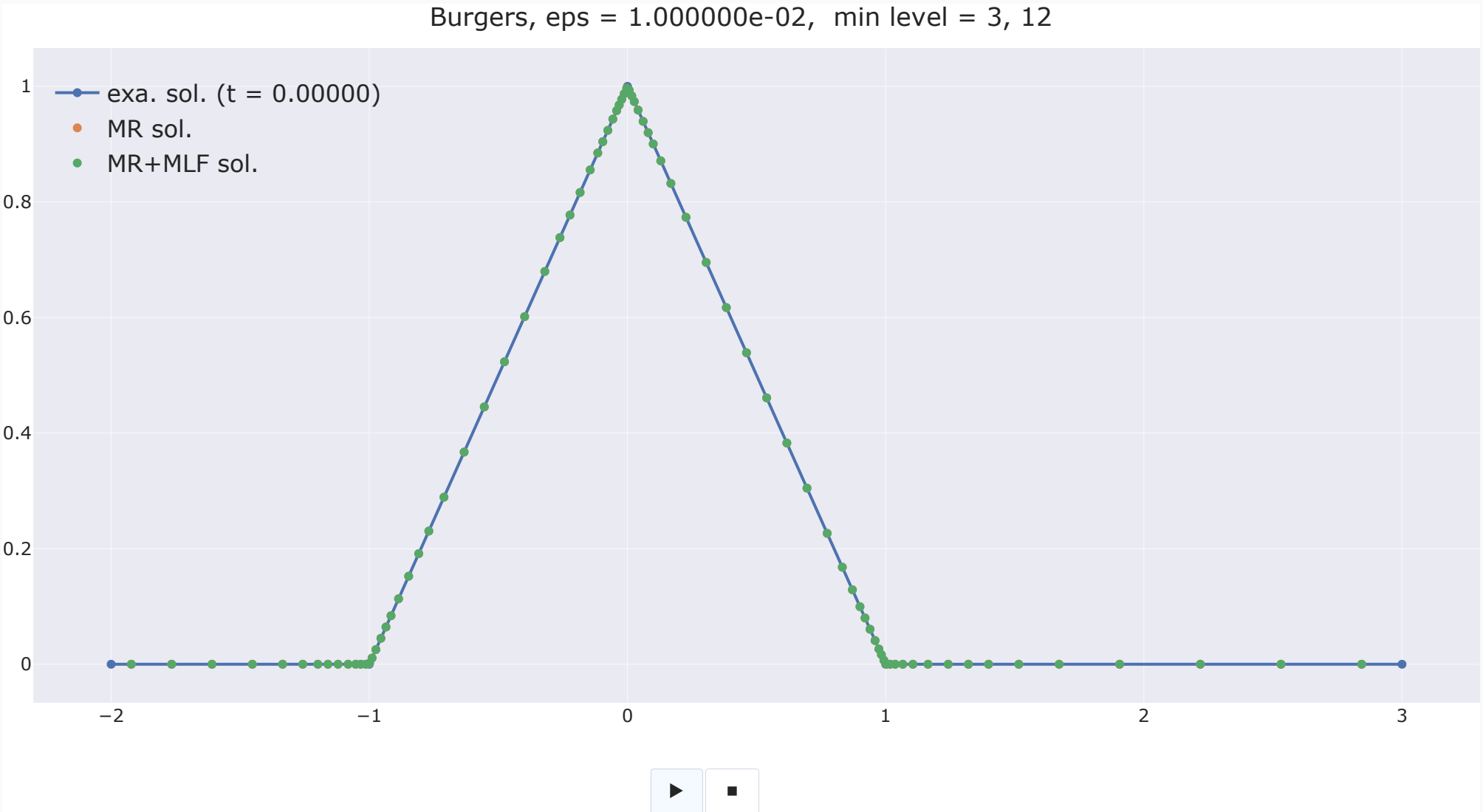
BURGERS RESULTS (MR+MLF SOL. AND LEVEL FOR SCHEME ORDER 3)



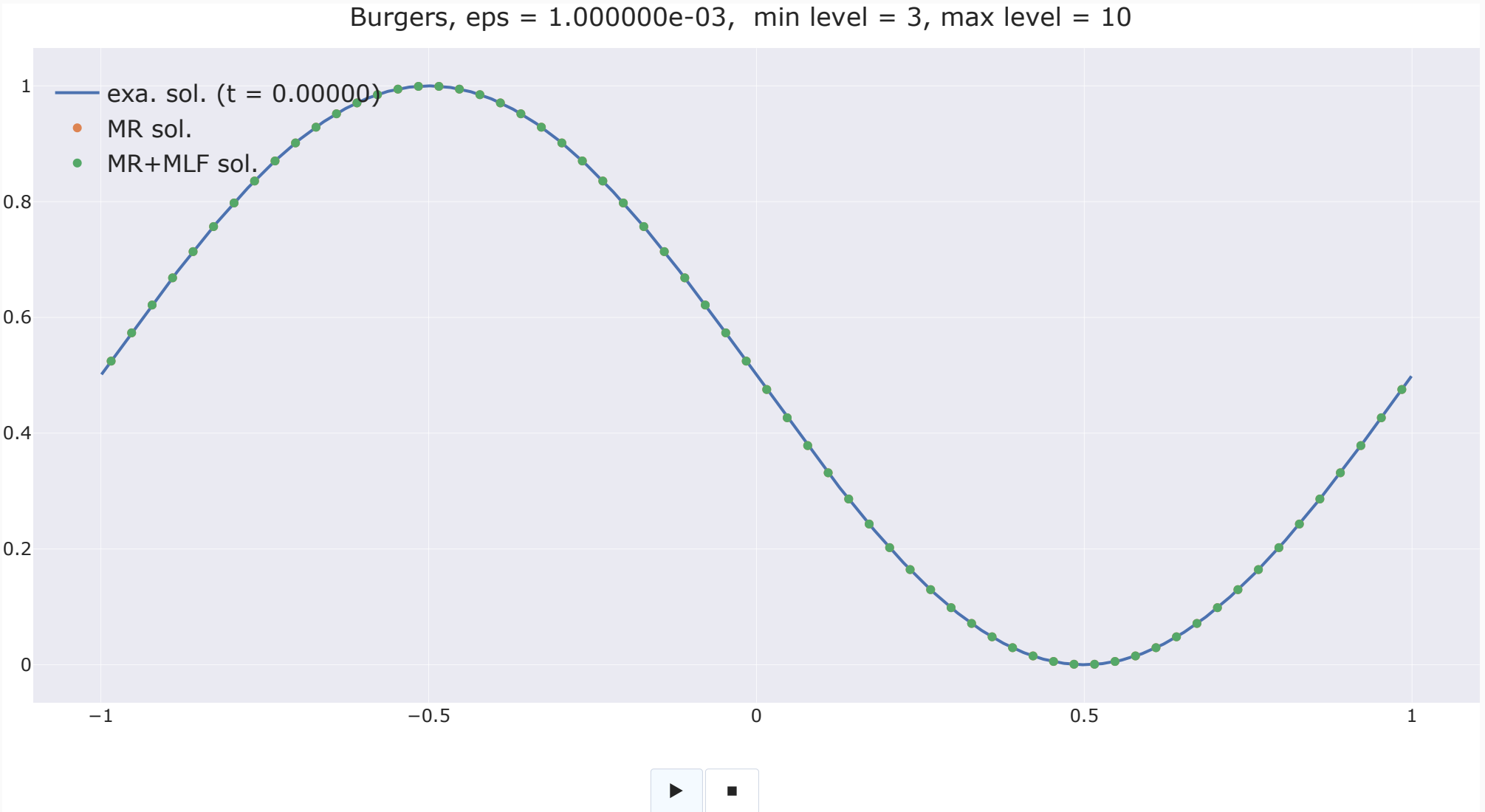
BURGERS RESULTS (MR+MLF SOL. AND ERROR FOR SCHEME ORDER 3)

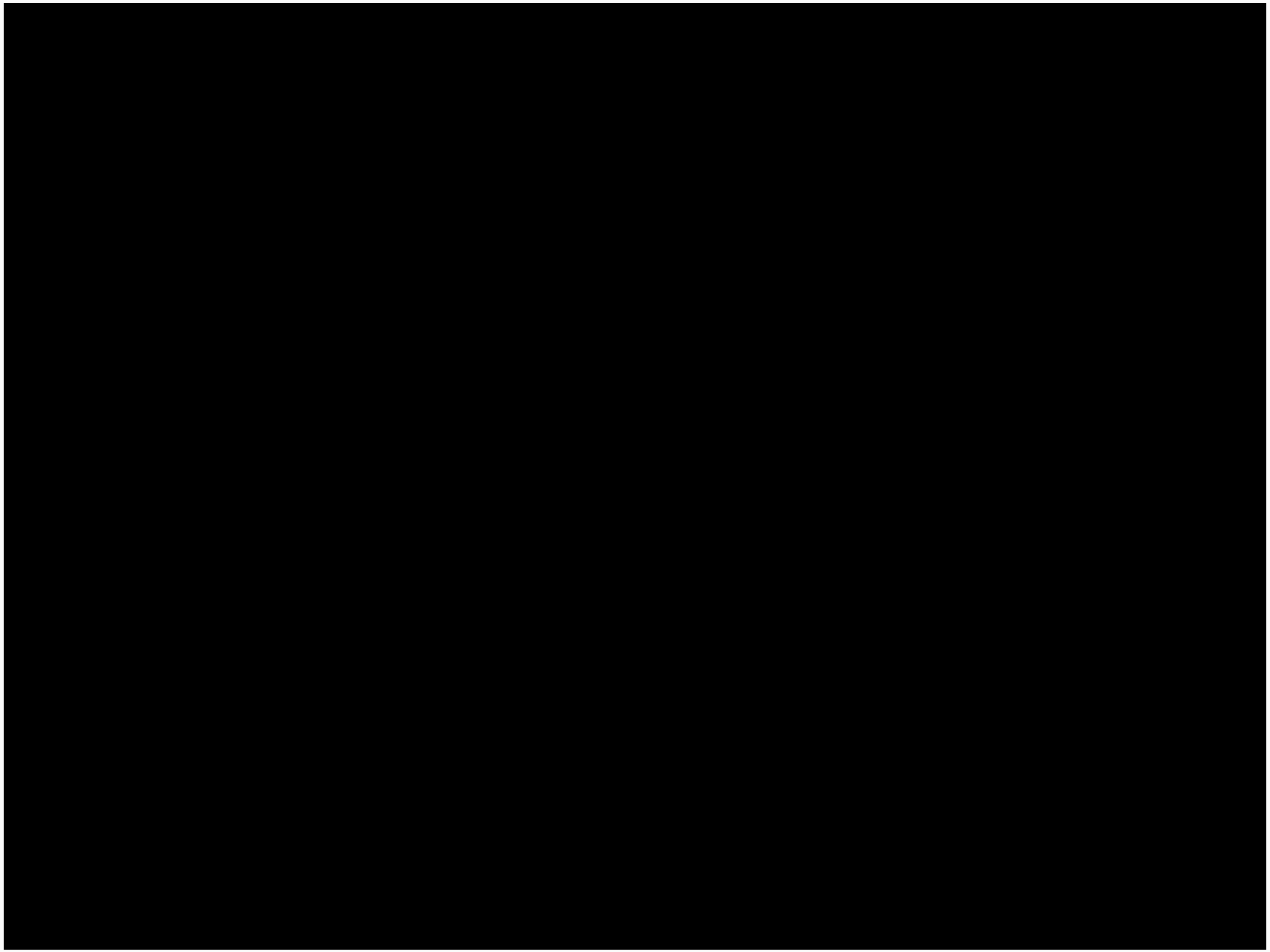


BURGERS RESULTS (MR+MLF SOL. ORDER 1) SMALL HAT



BURGERS RESULTS (MR+MLF SOL. ORDER 1) SINUS





$\epsilon = 1e-2, \underline{\ell} = 2, \bar{\ell} = 12$