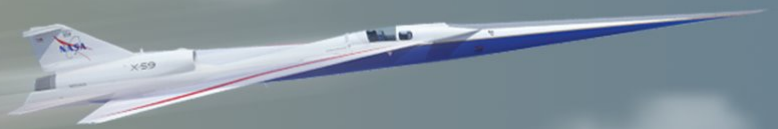
Recent Enhancements to Modeling Sonic Boom Propagation using Augmented Burgers' Equation

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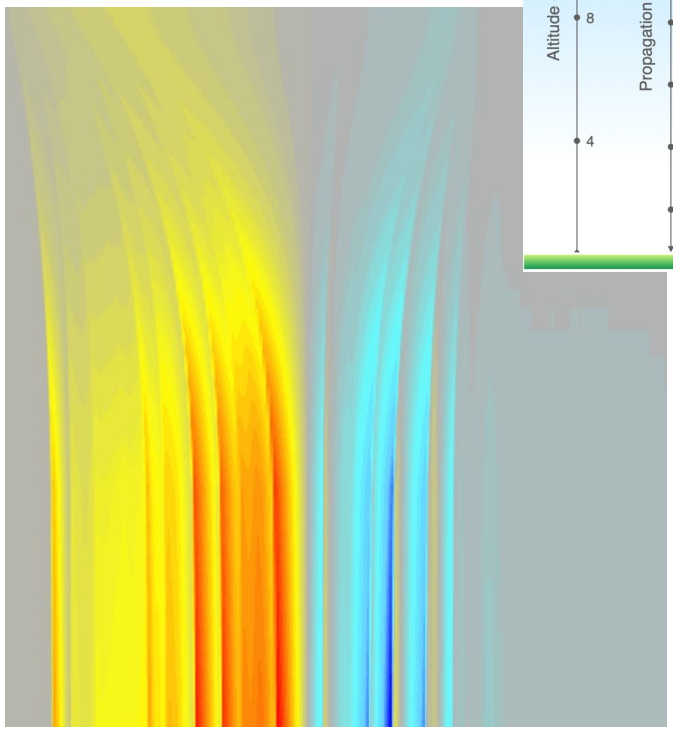
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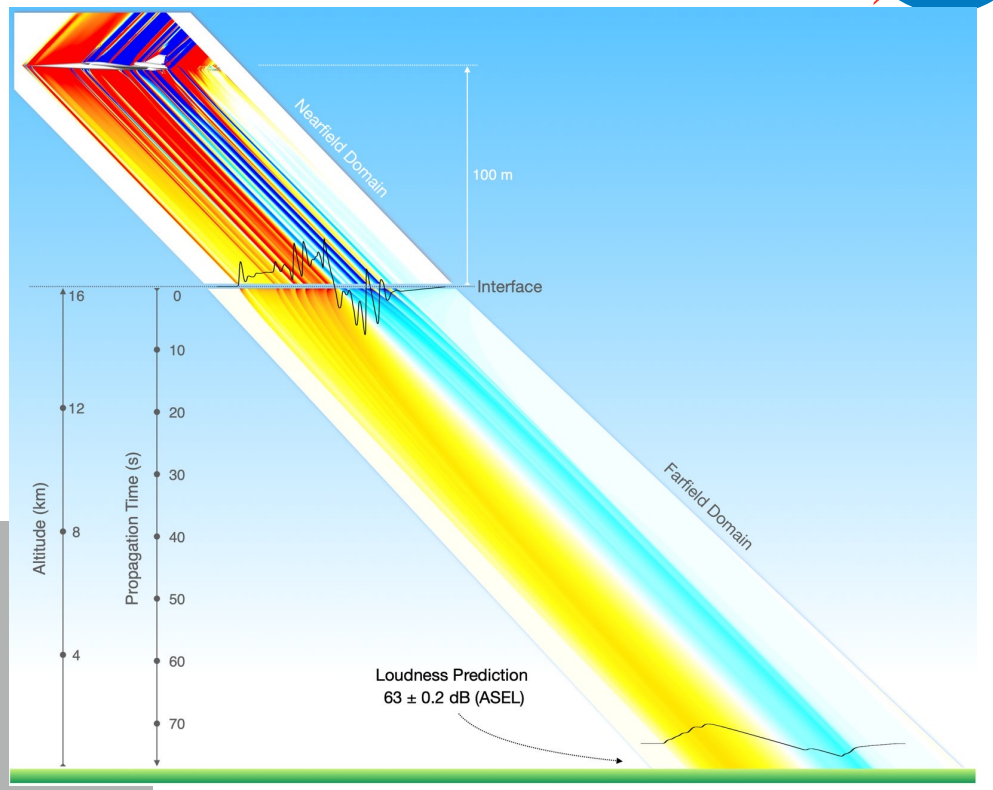


Advanced Modeling & Simulation (AMS) Seminar Series
NASA Ames Research Center, July 27th, 2023

Outline

- Introduction
  - Modeling & Simulation
- Implementation Updates
  - Non-linearity
  - Relaxation
- sBOOM
  - Primal
  - Dual (Adjoints)
- Loudness Metrics & Sensitivities
- Results
  - Simple Demonstration Case
  - X-59 Loudness Analysis
- Concluding Remarks





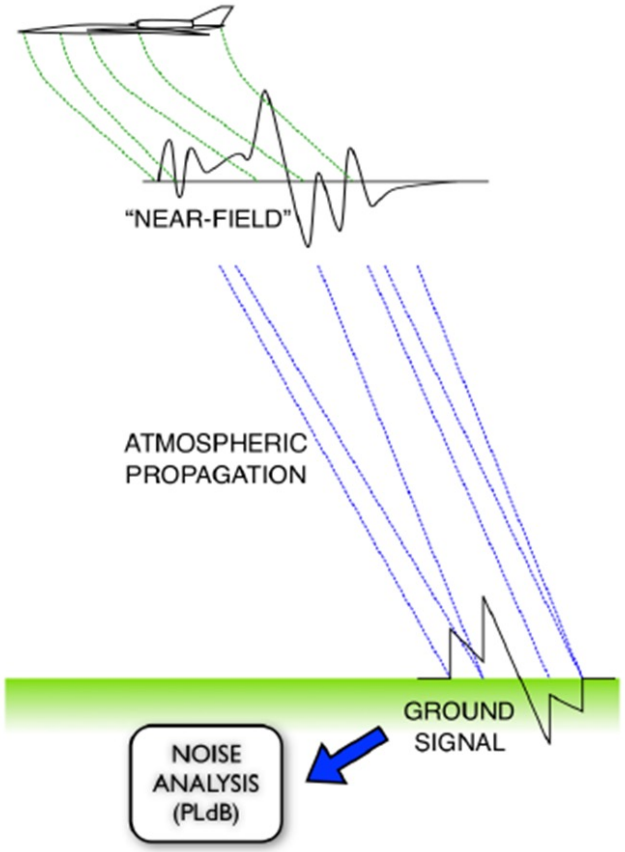
Sonic Boom Introduction



Disturbances from supersonic aircraft propagating to generate a sonic boom on the ground

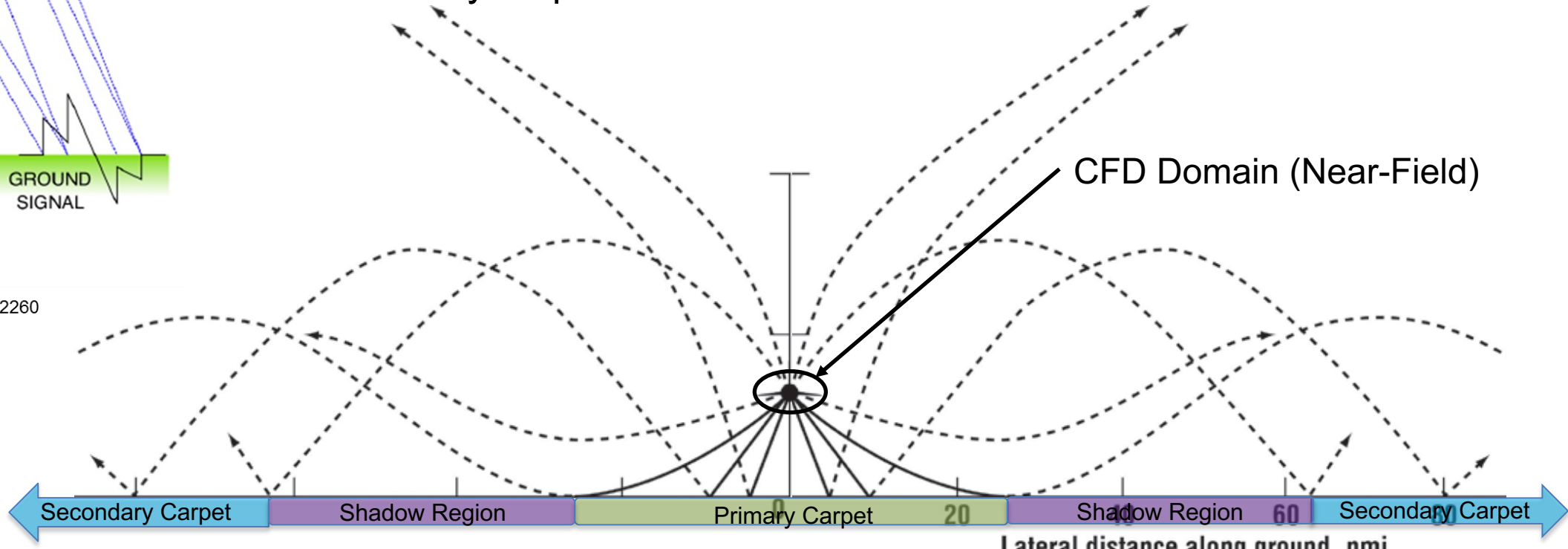
Primary Sonic Boom Carpet (Sweeping Parabolic Region) on the Ground

Sonic Boom Computation and Ray Paths



Wintzer AIAA 2015-2260

- Sonic boom computation involves
  - Near-field pressures from CFD
  - Atmospheric propagation to the ground
- Aircraft disturbances propagated along ray paths
  - Rays go in all directions
  - Can separate impact into primary carpet, shadow region and secondary carpet



Sonic Boom Ray Path Equations

- Ray paths dictated by coupled differential equations
  - Equally applicable to both primary and secondary booms
- Under windless conditions, equations can be simplified

$$\frac{d\vec{x}}{dt} = \vec{w} + c^2\vec{q}D$$

$$\frac{d\vec{q}}{dt} = -\left[\frac{\nabla c}{cD} + \sum_{i=1}^{3} q_i \nabla w_i\right]$$

$$\vec{q} = \frac{\hat{n}}{c + \vec{w}.\hat{n}}$$

$$D = 1 + \frac{\vec{w}.\hat{n}}{c}$$

Zero Winds

$$\frac{d\vec{x}}{dt} = c^2\vec{q}$$

$$\frac{d\vec{q}}{dt} = -\frac{\nabla c}{c}$$

$$\vec{q} = \frac{\hat{n}}{c}$$

Boom Propagation Modeling: Augmented Burgers Equation

- Most sonic boom propagation models based on numerical solution of 1-D augmented Burgers Equation

$$\frac{\partial p}{\partial \sigma} = \frac{\frac{\partial B}{\partial \sigma}}{B} p + \frac{\beta \Omega}{2\rho_0 c_0^3}\left(\frac{c_0}{v_g}\right)\frac{\partial p^2}{\partial \tau} + \frac{\delta \Omega^2}{2\rho_0 c_0^3}\left(\frac{c_0}{v_g}\right)\frac{\partial^2 p}{\partial \tau^2} + \sum_\nu \frac{m_\nu}{2c_0}\frac{t_\nu \Omega^2}{1+t_\nu \Omega \frac{\partial}{\partial \tau}}\left(\frac{c_0}{v_g}\right)\frac{\partial^2 p}{\partial \tau^2}$$

Molecular Relaxation

Nonlinearity

Thermoviscous absorption

$$B = \sqrt{\frac{\rho c S_0 D_0^2}{\rho_0 c_0 S D^2}\left(\frac{\frac{c v_{g0}}{D_0}}{\frac{c_0 v_g}{D}}\right)}, v_g = |c.\bar{n} + \bar{w}|, \mathrm{D} = 1 + \frac{\bar{w}.\bar{n}}{c}, \Omega = \frac{1}{D}$$

- D = Doppler wind effects

Numerical Solution Strategy

- Numerical solution implementation generally follows non-dimensionalization and operator splitting
- Non-dimensional version of augmented Burgers' equation

$$\frac{\partial P}{\partial \sigma} = P\frac{\partial P}{\partial \tau} + \frac{1}{\Gamma}\frac{\partial^2 P}{\partial \tau^2} + \sum_{\nu} C_{\nu}\frac{\frac{\partial^2}{\partial \tau^2}}{1+\theta_{\nu}\frac{\partial}{\partial \tau}}P - \frac{\frac{\partial A}{\partial \sigma}}{2A}P + \frac{\frac{\partial(\rho_0 c_0)}{\partial \sigma}}{2\rho_0 c_0}P$$

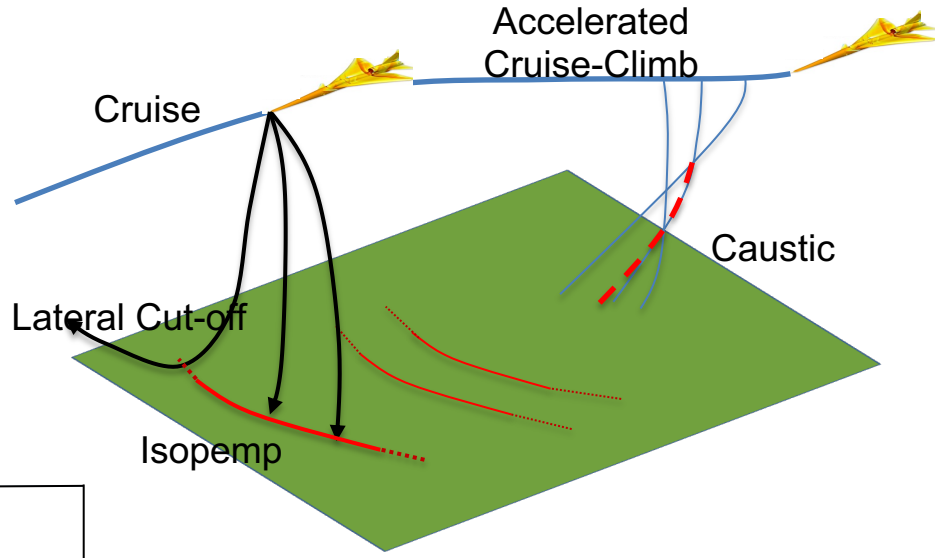
- Solution process involves operator splitting scheme

$$\begin{aligned}
\frac{\partial P}{\partial \sigma} &= p\frac{\partial P}{\partial \tau}\text{[Non-Linearity]} \\
\frac{\partial P}{\partial \sigma} &= \frac{1}{\Gamma}\frac{\partial^2 P}{\partial \tau^2}\text{[Absorption]} \\
\frac{\partial P}{\partial \sigma} &= \Sigma_{\nu}\frac{C_{\nu}\frac{\partial^2}{\partial \tau^2}}{1+\theta_{\nu}\frac{\partial}{\partial \tau}}P\text{[Molecular Relaxation]} \\
\frac{\partial P}{\partial \sigma} &= -\frac{1}{2G}\frac{\partial G}{\partial \sigma}P\text{[Ray Tube Spreading]} \\
\frac{\partial P}{\partial \sigma} &= \frac{1}{2\rho_0 c_0}\frac{\partial(\rho_0 c_0)}{\partial \sigma}P\text{[Atmospheric Stratification]}
\end{aligned}$$

[1]*Cleveland, R. O., "Propagation of Sonic Booms Through a Real, Stratified Atmosphere", PhD Thesis, UT Austin, May 1995*

sBOOM – Nonlinear Propagation Analysis and Design Tool

- Follows sonic boom propagation best practices
  - Ignores diffraction (near the carpet edges)
- Augmented Burgers' equation validated with acoustic measurements with fairly good agreement



$$\frac{\partial p}{\partial \sigma} = \frac{\beta\Omega}{2\rho_0 c_0^3}\left(\frac{c_0}{v_g}\right)\frac{\partial p^2}{\partial \tau}$$ *Nonlinearity*

$$+\frac{\partial p}{\partial \sigma}\frac{\frac{\partial B}{\partial \sigma}}{B}p$$ *Blokhintzev Invariance*

$$+\frac{\delta\Omega^2}{2\rho_0 c_0^3}\left(\frac{c_0}{v_g}\right)\frac{\partial^2 p}{\partial \tau^2}$$ *Absorption*

$$+\sum_\nu \frac{m_\nu}{2c_0}\frac{t_\nu\Omega^2}{1+t_\nu\Omega\frac{\partial}{\partial\tau}}\left(\frac{c_0}{v_g}\right)\frac{\partial^2 p}{\partial \tau^2}$$ *Molecular relaxation*

$$A_n q_n = k_n B_n p_{n-1}$$

$$A_{n,2} r_n = B_{n,2} q_n$$

$$A_{n,3} t_n = B_{n,3} r_n$$

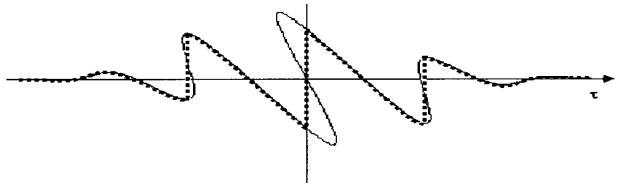
$$p_n = f(t_n)$$

Previous stable release of sBOOM v2.8.6

Implementation of Nonlinearity

- Non-linear term

$$\frac{\partial p}{\partial \sigma} = \frac{\beta \Omega}{2\rho_0 c_0^3}\left(\frac{c_0}{v_g D}\right)\frac{\partial p^2}{\partial \tau}$$



Source: Marchiano, R., Coulouvrat, F., and Grenon, R., "Numerical Simulation of Shock Wave Focusing at Fold Caustics, with Application to Sonic Boom," The Journal of the Acoustical Society of America, Vol. 114, No. 4, 2003, pp. 1758–1771.doi:10.1121/1.1610459.

- After non-dimensionalization

$$\frac{\partial P}{\partial \sigma} = P\left(\frac{c_0}{v_g D}\right)\frac{\partial P}{\partial \tau}$$

- Solve using the Poisson solution

$$P(\sigma + \Delta\sigma, \tau) = P(\sigma, \tau + P\left(\frac{c_0}{v_g D}\right)\Delta\sigma)$$

- The waveform is mapped to a distorted grid and re-interpolated
- Multi-valuedness is avoided using:

$$\Delta\sigma < \frac{1}{\max(\frac{\partial P}{\partial \tau})}$$

sBOOM v 2.8.6

Gudunov's Flux Function

$$P_i'^n = P_i^n - \frac{\Delta\sigma}{\Delta\tau}R_i$$

$$P_i^{n+1} = \frac{1}{2}(P_i^n + P_i'^n) - \frac{\Delta\sigma}{2\Delta\tau}R_i'$$

$$R_i = -F(U_{L,i-1}, U_{R,i}) + F(U_{L,i}, U_{R,i+1})$$

- Conservative finite-volume discretization using Gudunov's flux function with explicit Euler time marching
- For stability, ratio of temporal and spatial step-sizes < 1.0
- 2nd order accurate compared to 1st order accuracy for v2.8.6

sBOOM v 2.9.0

Implementation of Absorption

- Absorption equation

$$\frac{\partial p}{\partial \sigma} = \frac{\delta \Omega^2}{2\rho_0 c_0^3}\left(\frac{c_0}{v_g}\right)\left(\frac{1}{D^2}\right)\frac{\partial^2 p}{\partial \tau^2}$$

- Non-dimensionalization results in the following equation

$$\frac{\partial P}{\partial \sigma} = \frac{1}{\Gamma}\left(\frac{c_0}{v_g}\right)\left(\frac{1}{D^2}\right)\frac{\partial^2 P}{\partial \tau^2}$$

Similar in sBOOM v 2.8.6 and sBOOM v2.9.0

- Central difference scheme used for RHS
  - Crank-Nicolson scheme used

$$\frac{\partial^2 P}{\partial \tau^2} = \frac{P(\tau + \Delta\tau) - 2P(\tau) + P((\tau - \Delta\tau)}{(\Delta\tau)^2} + O[(\Delta\tau)^2]$$

- First order forward difference scheme used for LHS

$$\frac{\partial P}{\partial \sigma} = \frac{P(\sigma + \Delta\sigma) - P(\sigma)}{\Delta\sigma} + O[\Delta\sigma]$$

- Results in a tridiagonal system, efficiently solved using Thomas Algorithm

Implementation of Relaxation

- Relaxation equation:

$$\frac{\partial p}{\partial \sigma} = \sum_{\nu} \frac{m_{\nu}}{2c_0} \frac{t_{\nu}\Omega^2}{1+\frac{t_{\nu}}{D}\Omega\frac{\partial}{\partial \tau}} \left(\frac{c_0}{v_g}\right)\left(\frac{1}{D^2}\right)\frac{\partial^2 p}{\partial \tau^2}$$

- Non-dimensionalization results in the following equation

$$\left(1+\frac{\theta_{\nu,orig}}{D}\frac{\partial}{\partial \tau}\right)\frac{\partial P}{\partial \sigma} = C_{\nu,orig}\left(\frac{c_0}{v_g}\right)\left(\frac{1}{D^2}\right)\frac{\partial^2 P}{\partial \tau^2}$$

- Forward-space, centered-time finite difference

$$\frac{\partial^2 P}{\partial \tau \partial \sigma} = \frac{P_{i+1}^{j+1} - P_{i-1}^{j+1} - \left(P_{i+1}^{j} - P_{i-1}^{j}\right)}{2\Delta\tau\Delta\sigma} + O[\Delta\sigma, \Delta\tau^2]$$

- Turns into a Thomas algorithm solution for a tri-diagonal system
  - Modified from original Cleveland Thesis

$$C_{\nu} = C_{\nu,orig}\left(\frac{c_0}{v_g}\right)\left(\frac{1}{D^2}\right) \qquad \theta_{\nu} = \theta_{\nu,orig}\left(\frac{1}{D}\right)$$

- Two relaxation mechanisms modeled: N_2 and O_2. CO_2 if needed

sBOOM v 2.8.6

$$\frac{\partial P}{\partial \sigma} = \Sigma_{\nu}\frac{\partial\left[C_{\nu}P_{\nu}\right]}{\partial \tau}$$

$$\frac{\partial\left[\theta_{\nu}(P_{\nu}-P)\right]}{\partial \tau} + P_{\nu} = 0$$

- Newer implementation decomposes into a two-step scheme with internal species pressures
  - Renders LHS of all mechanisms to be similar

Updates to the relaxation implementation improves the well-posedness and robustness of the propagation schemes

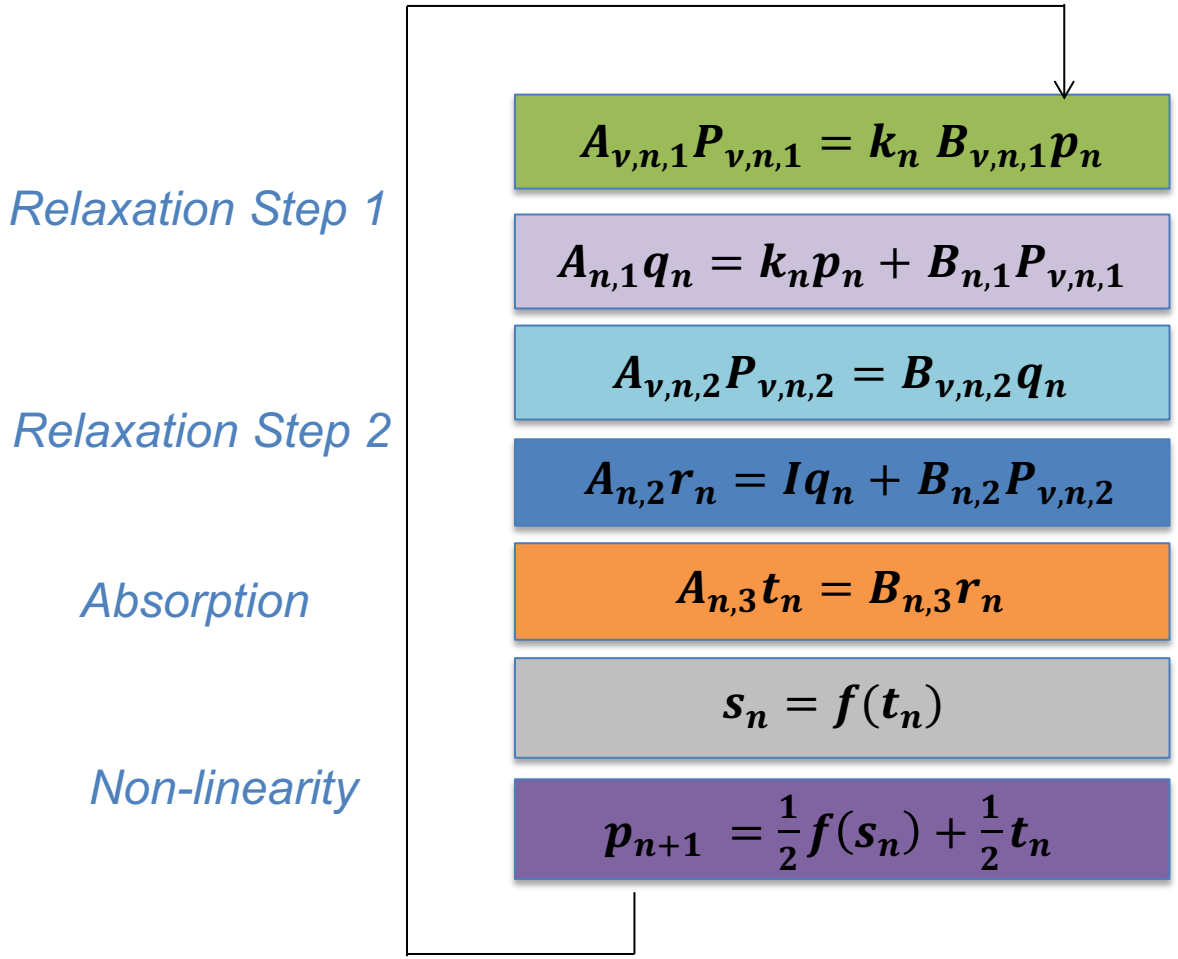
[1] Yamamoto, Hashimoto, Aoyama, et al. "A unified approach to an augmented Burgers equation for the propagation of sonic booms", The Journal of the Acoustical Society of America 137, 1857 (2015); doi: 10.1121/1.4916833

sBOOM v 2.9.0

sBOOM v2.9.0

- Propagation procedure involves iterative solution of a set of 7 equations

- 2nd order accurate numerical scheme in the spatial direction, 1st order accurate in the temporal domain



New stable release of sBOOM v2.9.0

Quantifying Impact of Sonic Booms

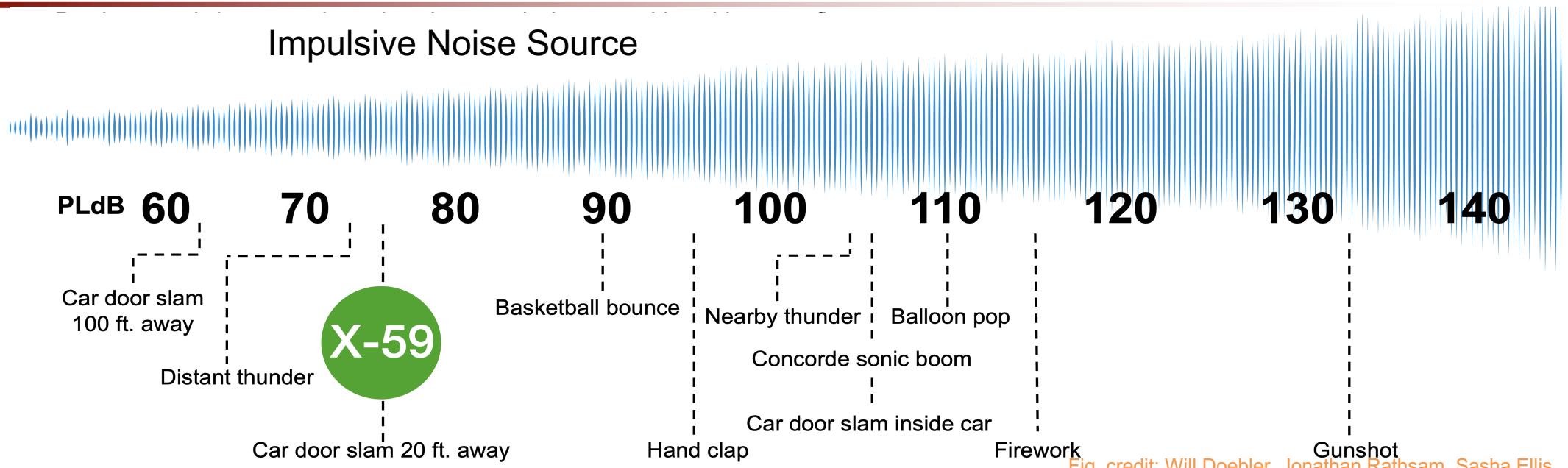
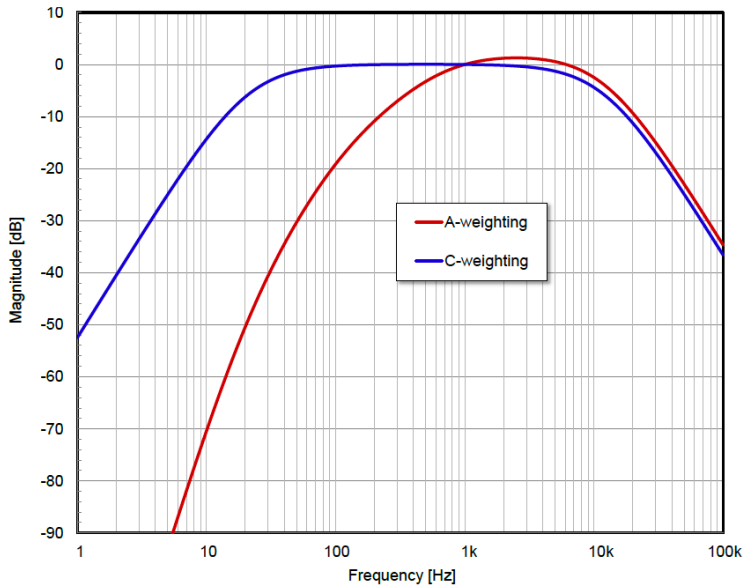
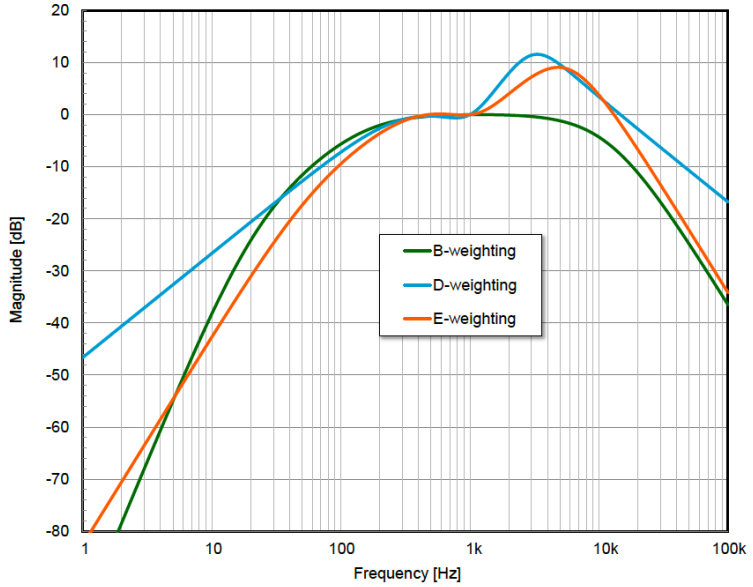


Fig. credit: Will Doebler, Jonathan Rathsam, Sasha Ellis
https://www.nasa.gov/sites/default/files/thumbnails/image/x-59_sonic_thump_75_pldb.png





Loudness Metrics using Digital Filters

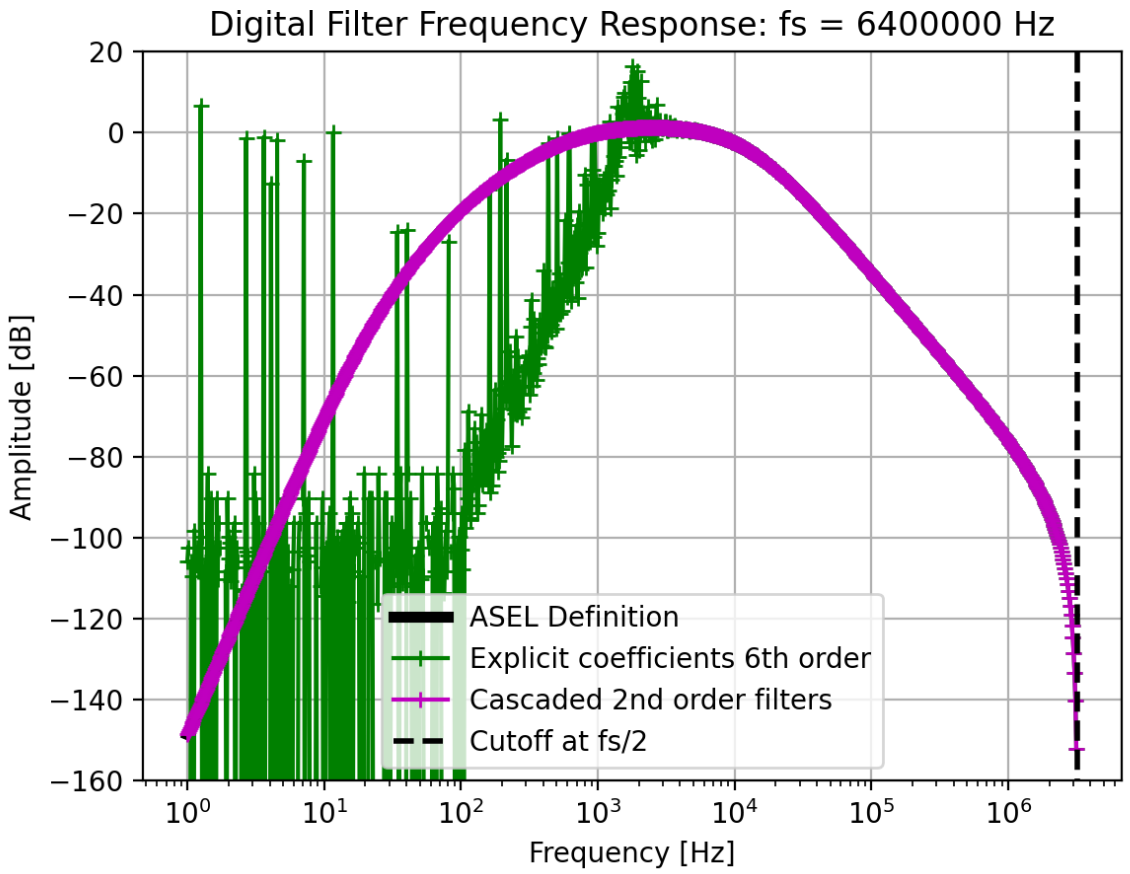
Loudness Metric Calculation:

- sBOOM uses digital filters as opposed to FFT's to compute sonic boom loudness metrics
- Digital filters from v2.8.6 were not well behaved at high sampling frequencies
- Updated sBOOM to represent higher-order polynomial transfer functions as a product of multiple quadratic transfer functions (Cascaded 2nd order filters) with an overall system gain

$$W(z) = \frac{B(z)}{A(z)} = \frac{\sum_{i=0}^{N} b_i z^i}{\sum_{i=0}^{M} a_i z^i}$$

$$W(z) = G \prod_{k=1}^{L} \frac{\sum_{i=0}^{2} b_{i,k} z^i}{\sum_{i=0}^{2} a_{i,k} z^i}$$

Fig. credit: Wade Spurlock, NASA ARC

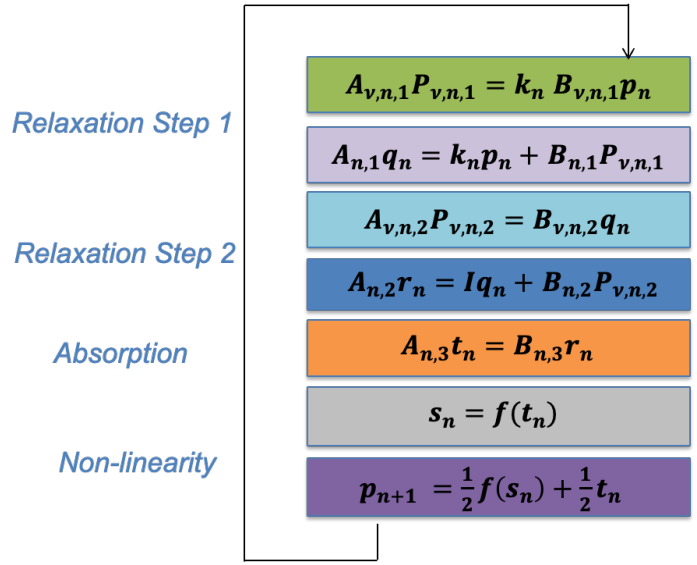


Updates to the loudness calculation improved robustness of metric predictions

Boom Adjoints

- sBOOM numerics:

k_n	-	Blokhintzev scaling term
$A_{n,1}, B_{n,1}, A_{v,n,1}, B_{v,n,1}$	-	N_2 Relaxation matrices
$A_{n,2}, B_{n,2}, A_{v,n,2}, B_{v,n,2}$	-	O_2 Relaxation matrices
$A_{n,3}, B_{n,3}$	-	Absorption matrices
$s, f(t_n)$	-	Nonlinear terms



- Define your independent variable vector *D*: Could be near-field pressures or atmospheric parameters
- Discrete-adjoint Lagrangian:

Cost functional — Step 1: Relaxation 1 — Step 2: Relaxation 1

$$L = J(p_N) + \sum_{n=1}^{N} \epsilon_n^T \left[A_{n,v,1}P_{n,v,1} - k_n B_{n,v,1}P_n\right] + \sum_{n=1}^{N} \gamma_{0,n}^T \left[A_{n,1}q_n - B_{n,1}P_{n,v,1} - k_n P_n\right]$$

Step 1: Relaxation 2 — Step 2: Relaxation 2

$$+ \sum_{n=1}^{N} \delta_n^T \left[A_{n,v,2}P_{n,v,2} - B_{n,v,2}q_n\right] + \sum_{n=1}^{N} \gamma_{1,n}^T \left[A_{n,2}r_n - B_{n,2}P_{n,v,2} - Iq_n\right]$$

$$+ \sum_{n=1}^{N} \beta_n^T \left[A_{n,3}t_n - B_{n,3}r_n\right] + \sum_{n=1}^{N} \lambda_{n,1}^T \left[s_n - f(t_n)\right] + \sum_{n=1}^{N} \lambda_{n,2}^T \left[p_{n+1} - \frac{1}{2}f(s_n) - \frac{1}{2}t_n\right]$$

Thermoviscous absorption — Step 1: Nonlinearity — Step 2: Nonlinearity

Boom Adjoints (2)

- Derivative of Lagrangian:

$$\frac{dL}{dD} = \frac{\partial J}{\partial p_N}\frac{\partial p_N}{\partial D} + \sum_{n=1}^{N} \epsilon_n^T \left[A_{n,\nu,1}\frac{\partial P_{n,\nu,1}}{\partial D} - k_n B_{n,\nu,1}\frac{\partial p_n}{\partial D} \right] + \sum_{n=1}^{N} \gamma_{0,n}^T \left[A_{n,1}\frac{\partial q_n}{\partial D} - B_{n,1}\frac{\partial P_{n,\nu,1}}{\partial D} - k_n \frac{\partial p_n}{\partial D} \right]$$
$$+ \sum_{n=1}^{N} \delta_n^T \left[A_{n,\nu,2}\frac{\partial P_{n,\nu,2}}{\partial D} - B_{n,\nu,2}\frac{\partial q_n}{\partial D} \right] + \sum_{n=1}^{N} \gamma_{1,n}^T \left[A_{n,2}\frac{\partial r_n}{\partial D} - B_{n,2}\frac{\partial P_{n,\nu,2}}{\partial D} - I\frac{\partial q_n}{\partial D} \right]$$
$$+ \sum_{n=1}^{N} \beta_n^T \left[A_{n,3}\frac{\partial t_n}{\partial D} - B_{n,3}\frac{\partial r_n}{\partial D} \right] + \sum_{n=1}^{N} \lambda_{n,1}^T \left[\frac{\partial s_n}{\partial D} - \frac{\partial f}{\partial t_n}\frac{\partial t_n}{\partial D} \right] + \sum_{n=1}^{N} \lambda_{n,2}^T \left[\frac{\partial p_{n+1}}{\partial D} - \frac{1}{2}\frac{\partial f}{\partial s_n}\frac{\partial s_n}{\partial D} - \frac{1}{2}\frac{\partial t_n}{\partial D} \right]$$

- For deterministic problems, coefficient matrices $A_{n,\nu,1}$ etc. are not dependent on intermediate pressures
- Grouping “like” items results in a set of adjoint equations

Boom Adjoints (3)

- Adjoint (dual) solution procedure involves iterative solution of a set of 7 equations
- Solution begins at the ground (end of primal solve) and proceeds backwards

$$\lambda_{n,2}^T = -\frac{\partial J}{\partial p_N} \text{ when n = N}$$

$$\lambda_{n,2}^T = \epsilon_n^T k_n B_{n,\nu,1} + k_n \gamma_{0,n}^T \text{ Otherwise}$$

$$\lambda_{n,1}^T = \frac{1}{2}\lambda_{n,2}^T \frac{\partial f}{\partial s_n}$$

$$\beta_n^T A_{n,3} = \lambda_{n,1}^T \frac{\partial f}{\partial t_n} + \frac{1}{2}\lambda_{n,2}^T$$

$$\gamma_{1,n}^T A_{n,2} = \beta_n^T B_{n,3}$$

$$\delta_n^T A_{n,\nu,2} = \gamma_{1,n}^T B_{n,2}$$

$$\gamma_{0,n}^T A_{n,1} = \delta_n^T B_{n,\nu,2} + \gamma_{1,n}^T$$

$$\epsilon_n^T A_{\nu,1} = \gamma_{0,n}^T B_{n,1}$$

Loudness Metric Sensitivities

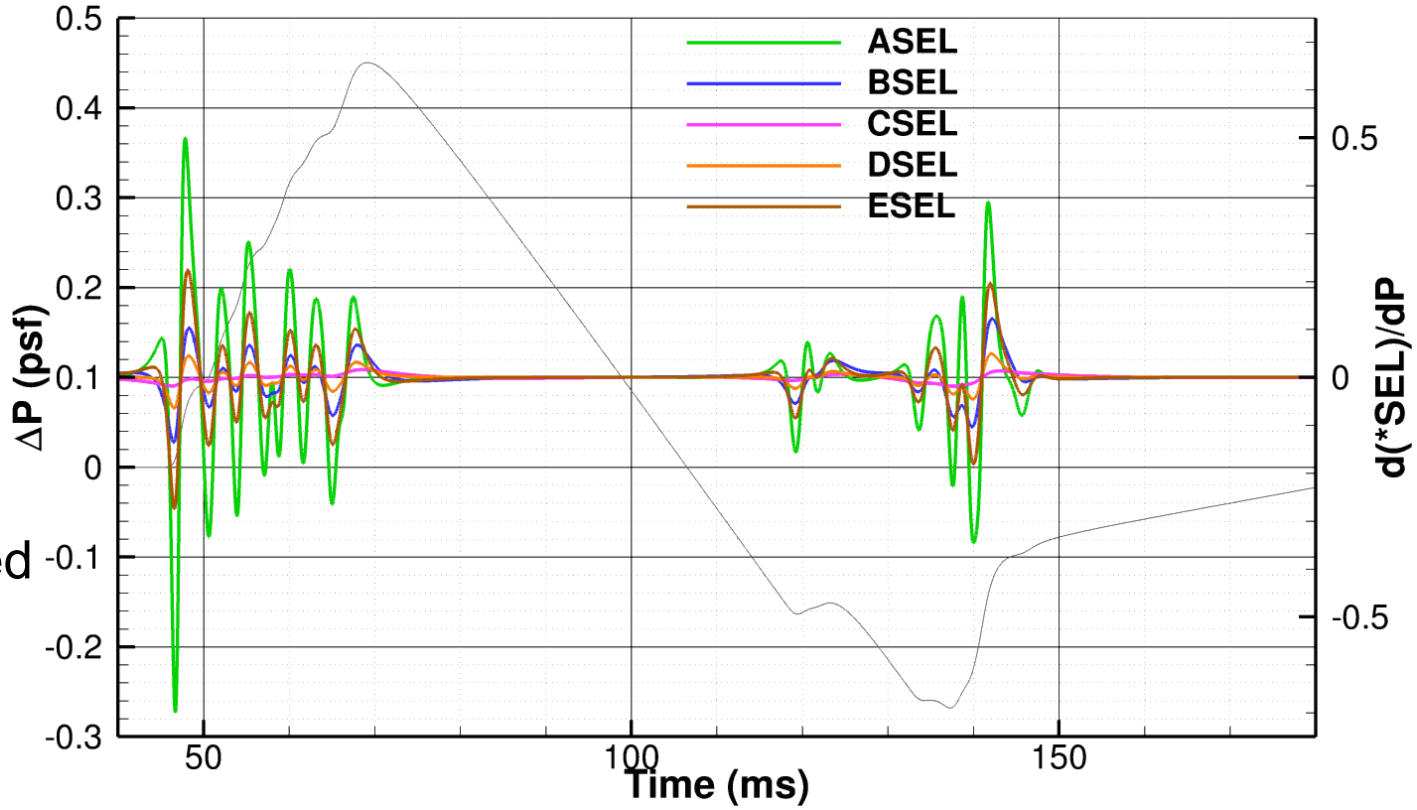
- Recall 2nd order cascaded filters

$$W(z) = \frac{B(z)}{A(z)} = \frac{\sum_{i=0}^{N} b_i z^i}{\sum_{i=0}^{M} a_i z^i}$$

$$W(z) = G \prod_{k=1}^{L} \frac{\sum_{i=0}^{2} b_{i,k} z^i}{\sum_{i=0}^{2} a_{i,k} z^i}$$

Loudness Metric Sensitivities:

- Cascaded filters analytically differentiated
- Sensitivities from different frequency weightings show smooth behavior

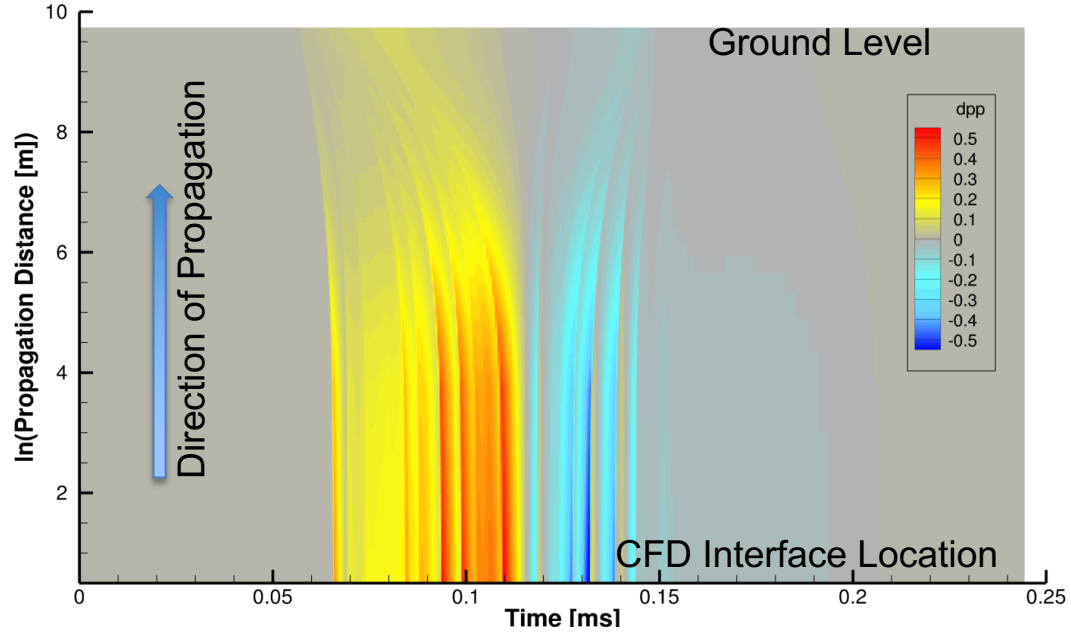


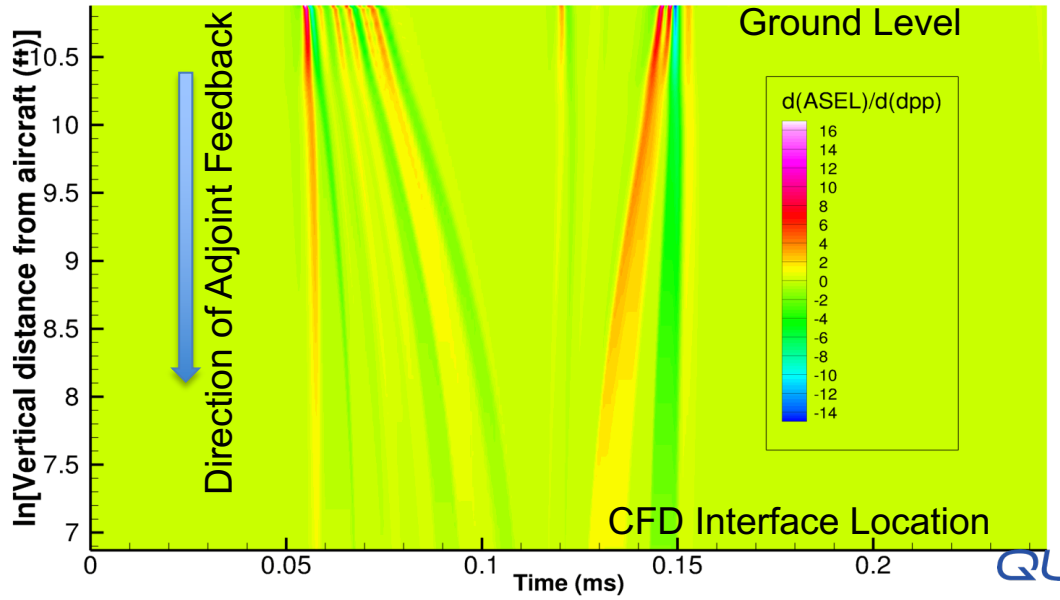
sBOOM Primal + Dual

- In addition to computing the sonic boom ground signatures, sBOOM provides gradients of the sonic boom ground pressures and loudness metrics
- Discrete adjoint equations solved for multiple cost functionals
- System of adjoint equations iteratively solved backwards using stored solutions from the primal solutions
  - Adjoint solutions initialized using the sensitivity of the cost functional at the ground

Adjoint gradients allow:

1. ***Critical feedback to aircraft designers for sonic boom mitigation***
2. ***Discrepancy investigation and troubleshooting between predictions and measurements***

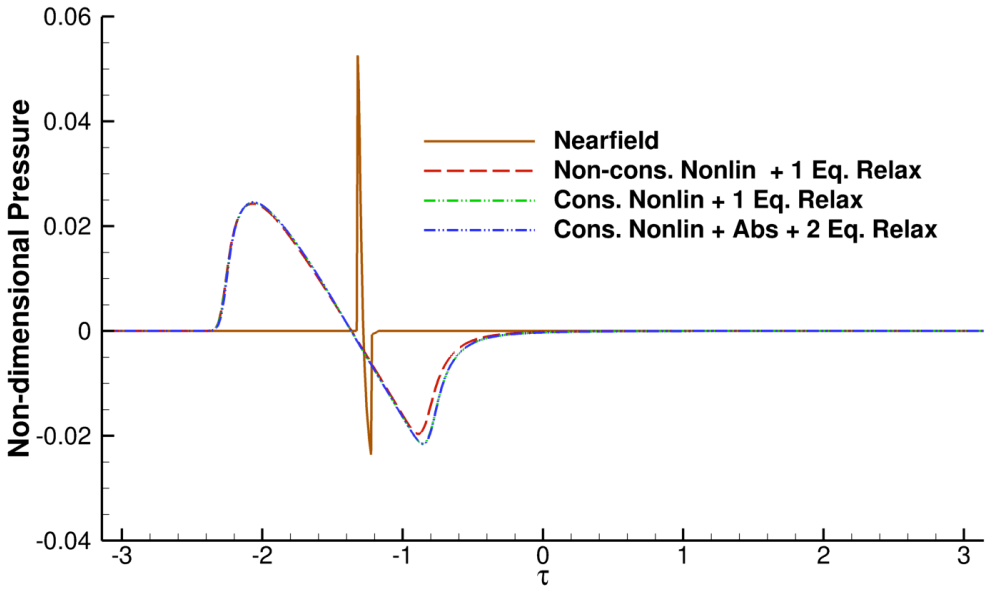


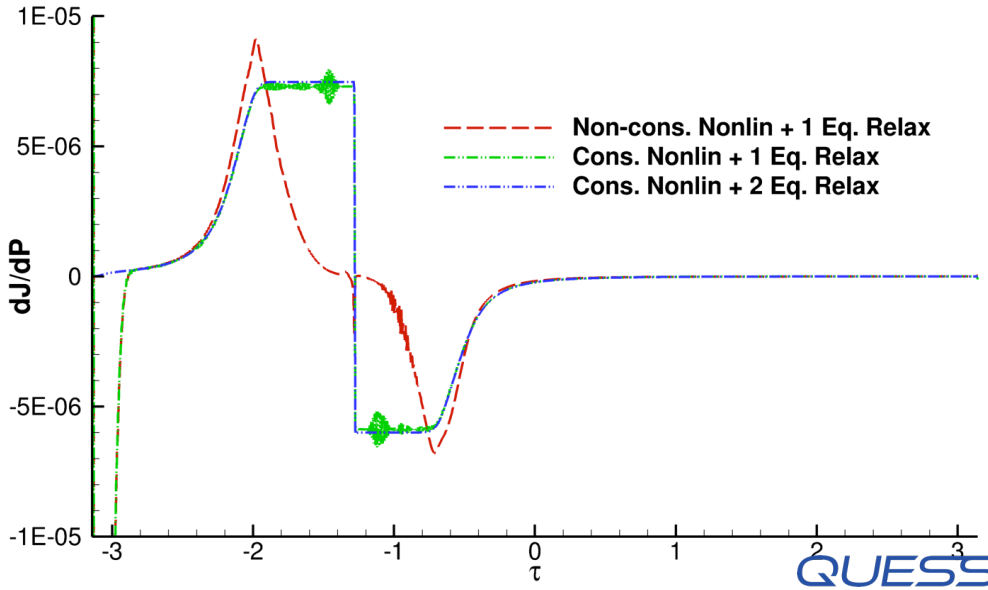


Results: Demonstration Case

- Simple demonstration case on a 2-shock system in the near-field, Mach = 1.4, Altitude = 40000 ft
- Primal solutions:
  - 1st order non-conservative nonlinearity shows differences in the aft compared to 2nd order conservative schemes
  - Almost identical results with mixed-derivative relaxation compared to 2 equation species relaxation
- Adjoint solutions:
  - 1st order non-conservative nonlinearity shows significant differences
  - Conservative nonlinearity improve adjoint solution character
    - Poor behavior at front BC
    - Numerical noise
  - 2 equation species relaxation combined with conservative nonlinearity improves adjoint solution behavior
    - Good boundary behavior
    - No numerical noise

Conservative nonlinearity combined with 2 equation species relaxation significantly improves adjoint solutions

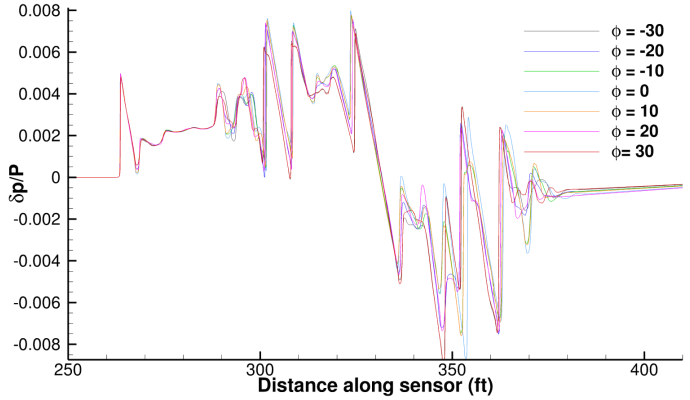




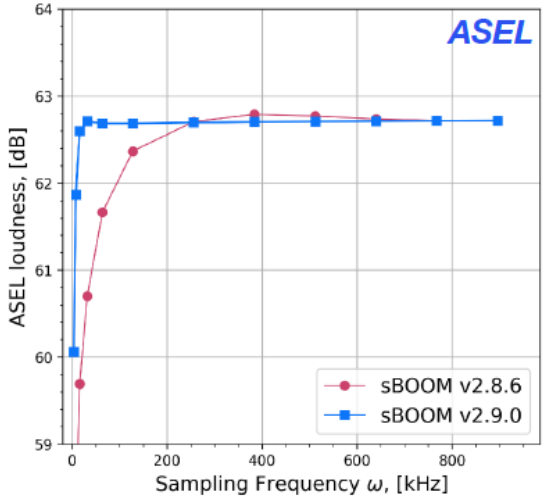
X-59: Loudness Convergence

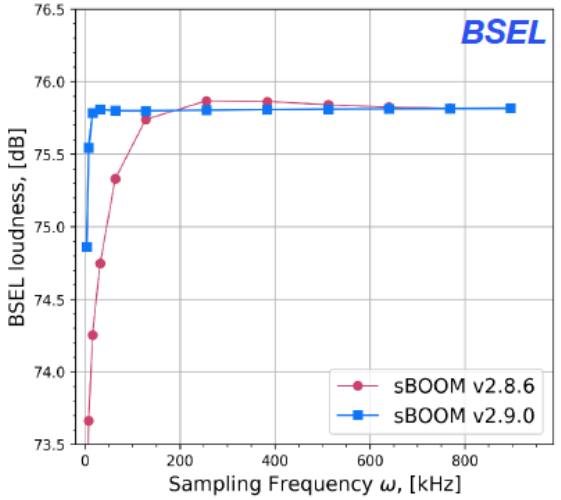
- X-59: C612A, baseline, level-cruise, altitude = 53200 ft in Std. Atm., φ = 0°

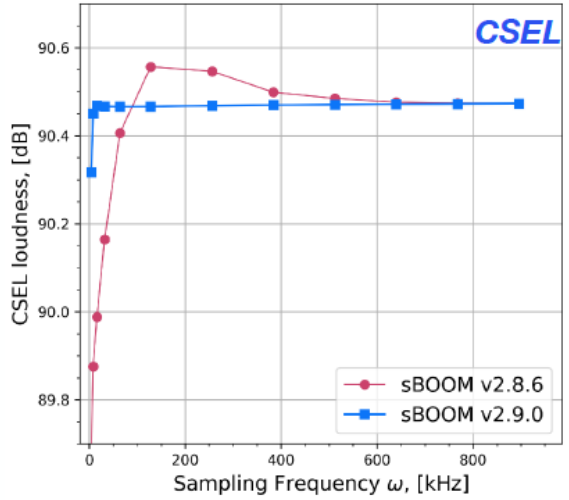


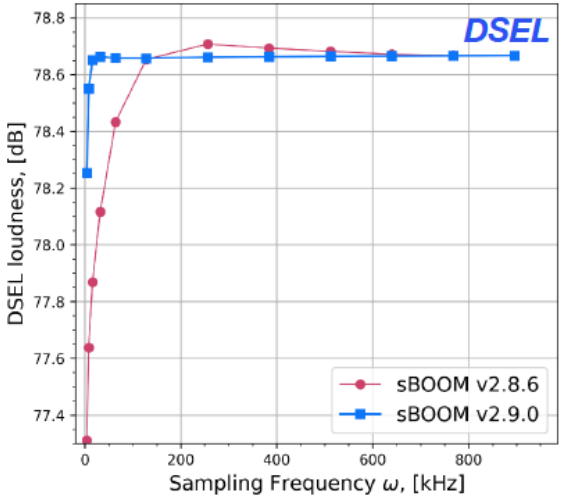


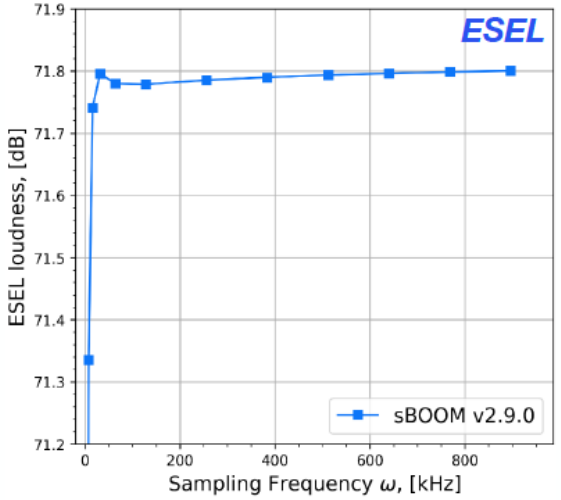
- Excellent loudness metric convergence for all loudness metrics

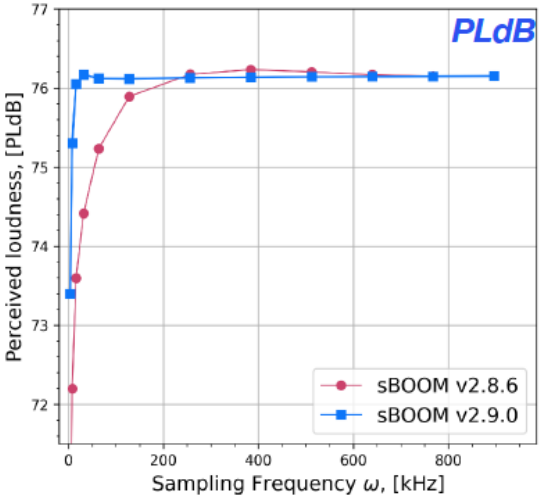








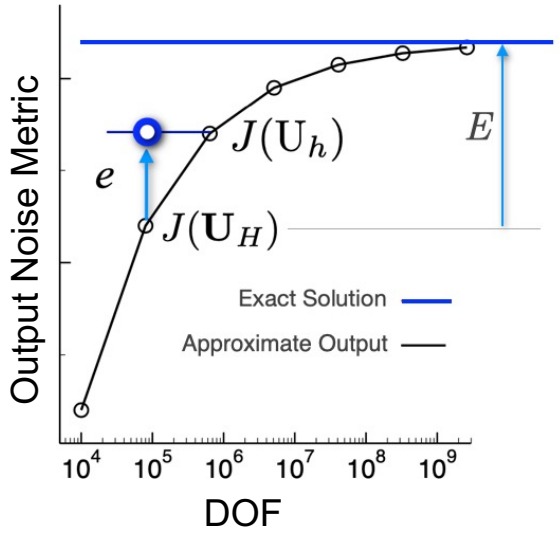




For similar output accuracy, sBOOM v2.9.0 is 8-12x faster than sBOOM v2.8.6

Results: Loudness Convergence Rate

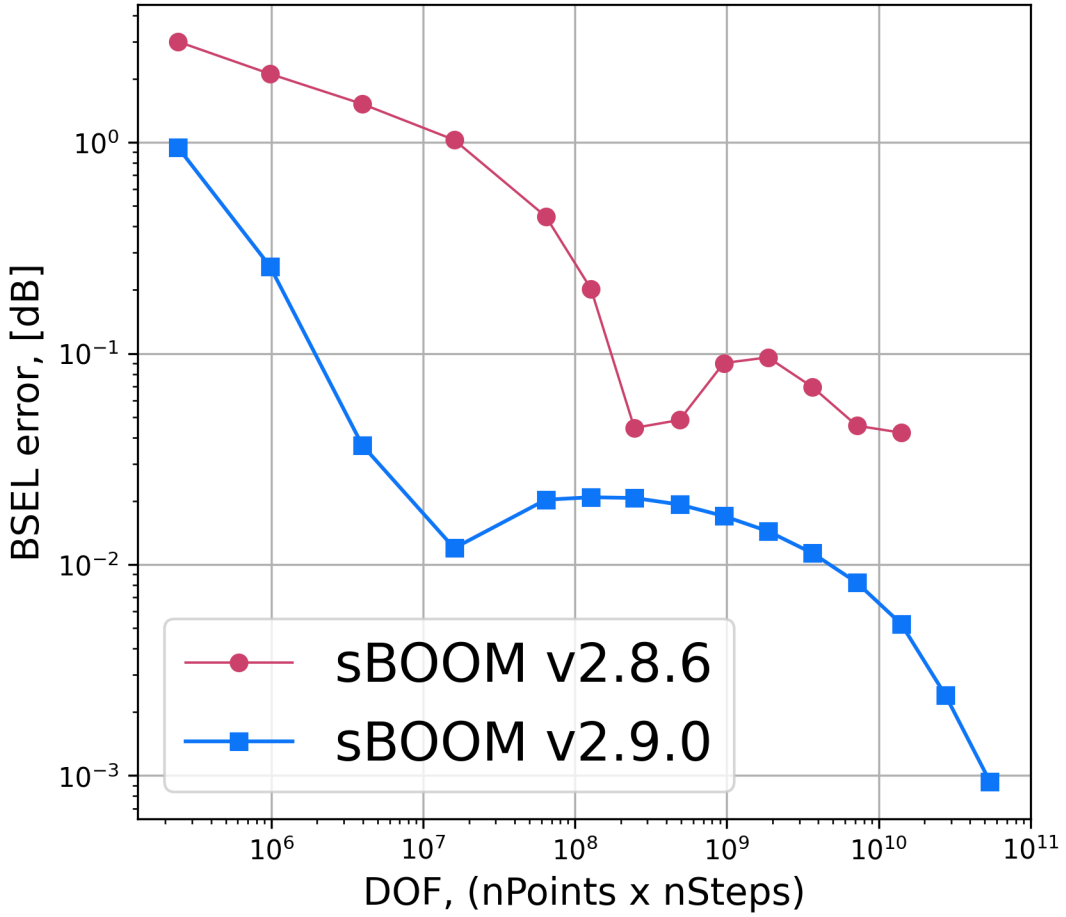
- Asymptotic error estimation and loudness convergence rates



$$E = e\left(1 + \frac{1}{K^2 - 1}\right)$$

K = Refinement ratio = H/h < 1

Convergence rate: $\alpha = \frac{log\frac{E(h_2)}{E(h_1)}}{log\frac{h_2}{h_1}}$



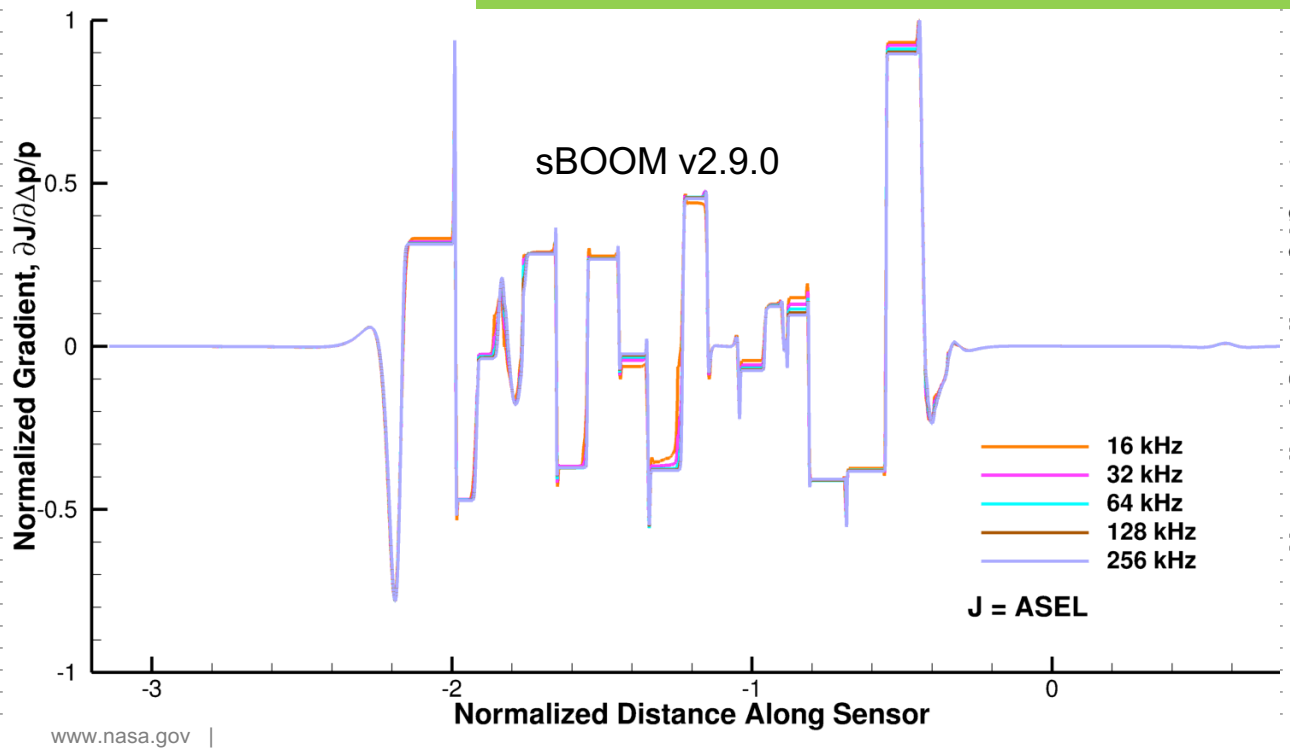
- Errors are higher and plateau using older sBOOM implementation
- Errors continue to drop with increase in degrees of freedom
  - DOF = nPoints x nSteps

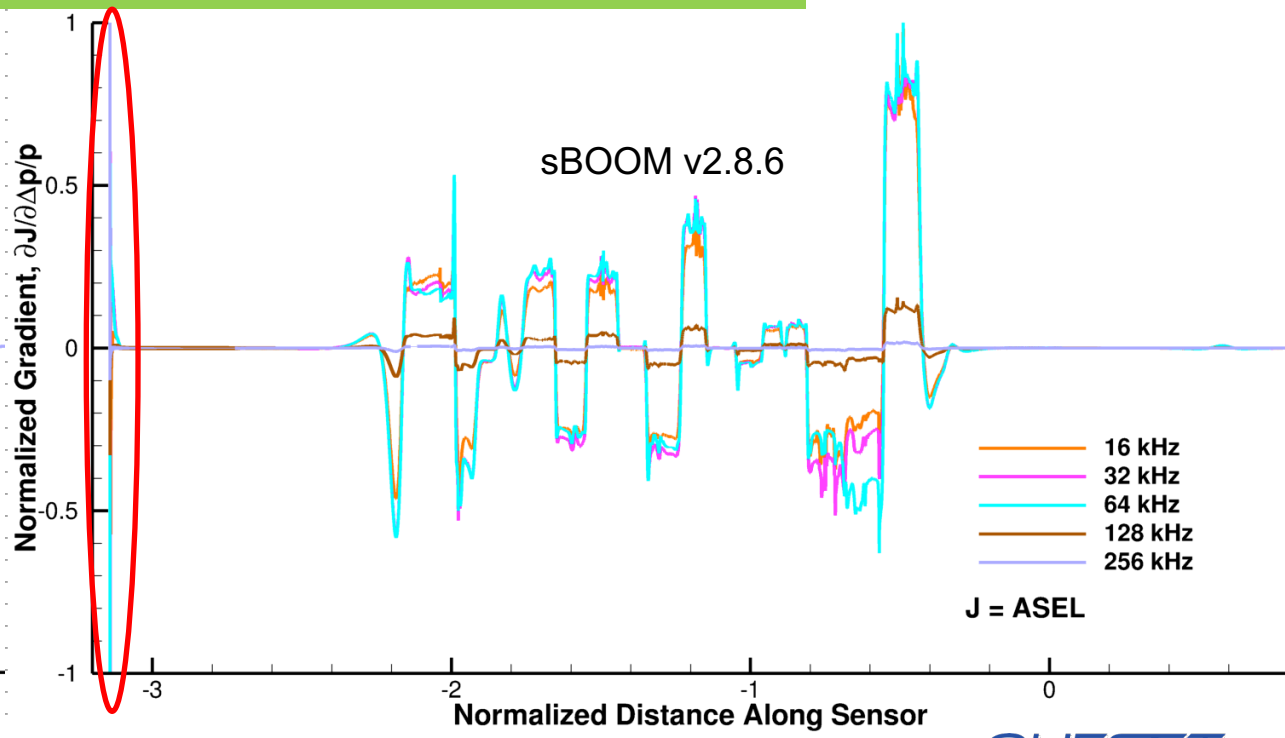
sBOOM 2.9.0 has lower errors and faster convergence of loudness metrics compared to sBOOM v2.8.6

Results: Convergence of Gradients

- ASEL gradients all converge very well for ASEL using sBOOMv2.9.0
  - Similar convergence for other adjoint cost functionals
- ASEL gradients from sBOOM v2.8.6
  - Noisy, but maintain same character below 64 kHz
  - Higher sampling frequencies result in error build-up, resulting in poor convergence behavior
  - Poor BC implementation

sBOOM v2.9.0 has adjoint gradients converging robustly

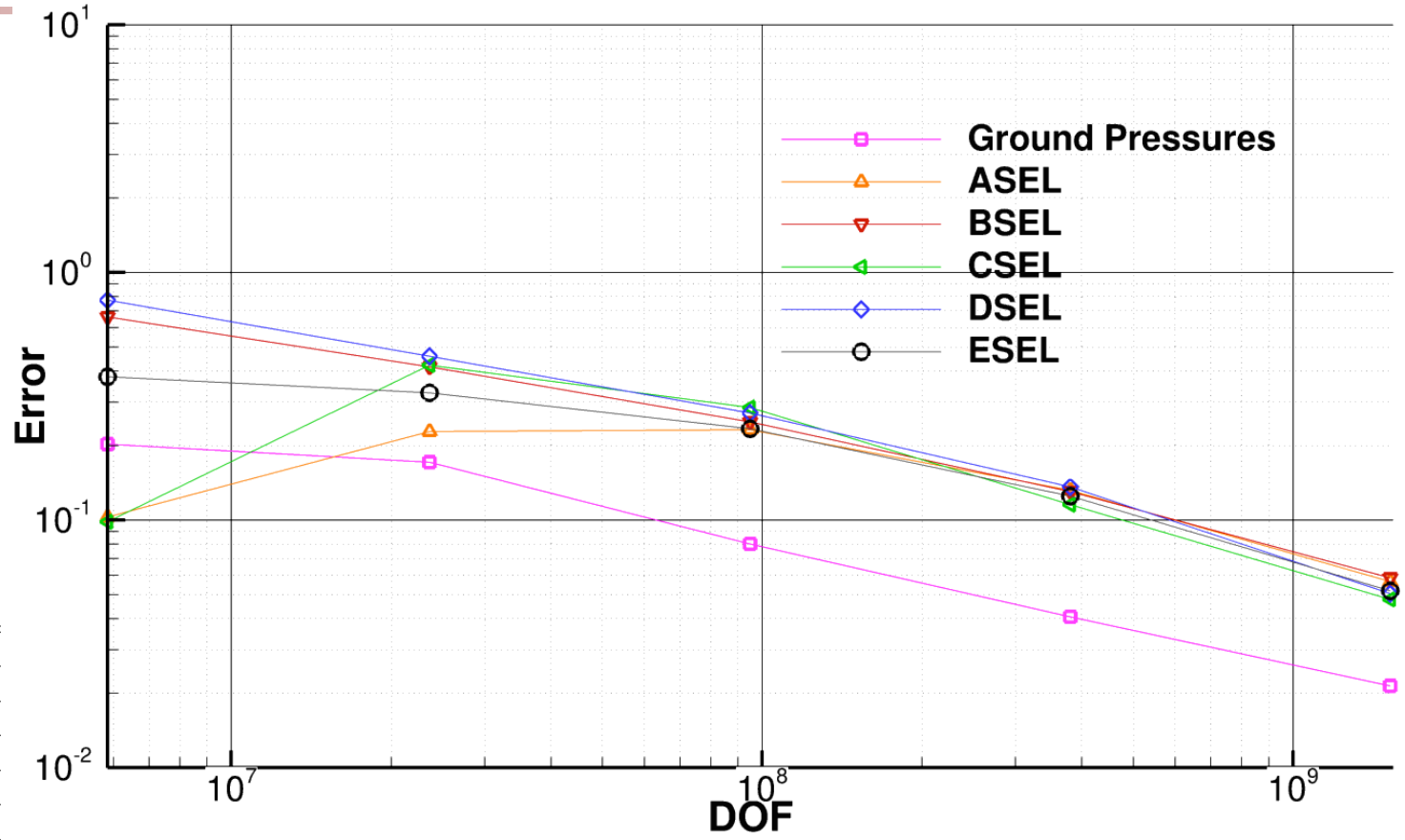




Results: Convergence of Gradients

- Order of accuracy of gradient convergence is ~ 0.6
  - Ground pressures and all loudness metrics show similar convergence rates for the adjoint gradients
  - Convergence rates affected by small errors and discontinuities in gradients

Cost functional	Order of Convergence (last 3 points)	Order of Convergence (last 2 points)
Ground Pressures	0.48	0.46
ASEL	0.50	0.61
BSEL	0.52	0.58
CSEL	0.64	0.63
DSEL	0.60	0.72
ESEL	0.54	0.63



sBOOM v2.9.0 adjoint gradient convergence in line with expectations

Concluding Remarks and Future Work

- Updated numerical discretization of the augmented Burgers' equation within sBOOM has been presented
  - Asymptotically consistent adjoint formulation
  - Faster convergence and reliable accuracy for both the primal and the dual solvers
- Sonic boom loudness metrics and their adjoint gradients shown to converge well
- Newly implemented updates result in an order of magnitude improvement in computational run times when compared against the previous stable version of sBOOM

Future Work

- Investigate discretization errors in operator splitting process and ray tracing
- Leverage CFD integration for effective and efficient shape optimization for boom minimization with other relevant constraints

Acknowledgements

- NASA Commercial Supersonic Technology (CST) project under Aeronautics Mission Directorate (ARMD)

- Wade Spurlock of Science and Technology Corporation (STC) for insights and inputs on digital filter implementation of loudness metric calculation

QUESSTIONS?