

A Variational Theory of Aero-hydrodynamics

Advanced Modeling & Simulation (AMS) Seminar Series

NASA Ames Research Center

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Motivation

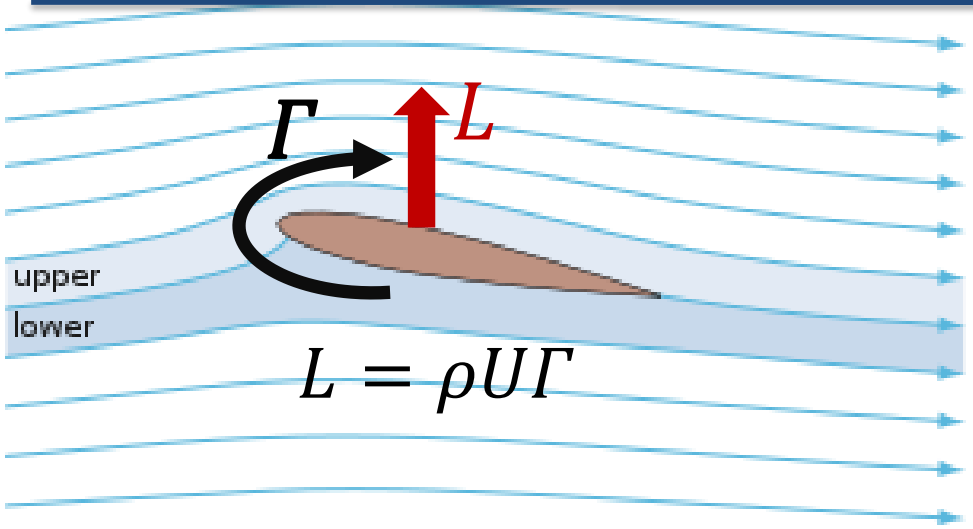
Kenneth Chang NY Times (2003):

"To those who fear flying, it is probably disconcerting that physicists and aeronautical engineers still passionately debate the fundamental issue underlying this endeavour: what keeps planes in the air?"

The Current Theory of Flight is Fragile

A New Theory of Lift

Non-uniqueness of Euler



Euler's Equation

$$\boldsymbol{u \cdot \nabla u = -\nabla p}$$

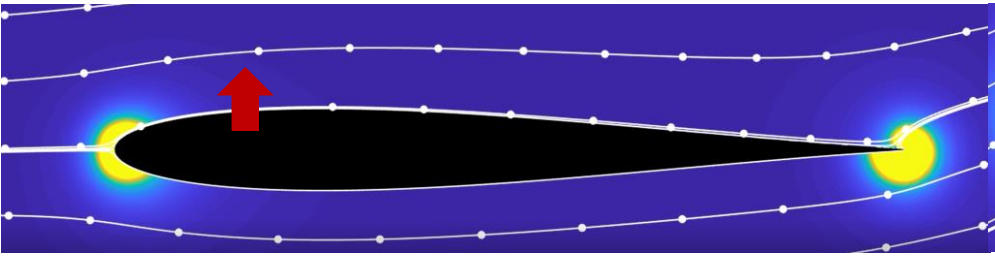
Given an irrotational solution u_0

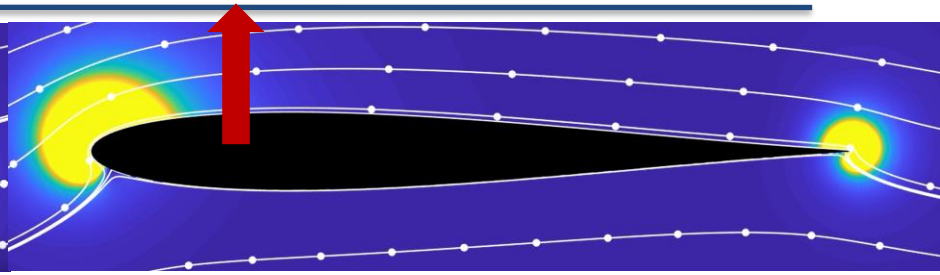
We can always construct u_1

$\boldsymbol{u_0 + \Gamma u_1}$ is a solution of the Euler Problem for any $\Gamma \in \mathcal{R}$

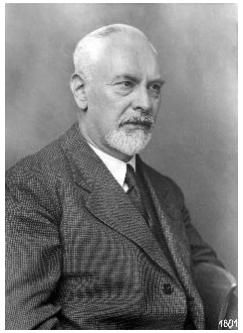
What solution does Nature select?

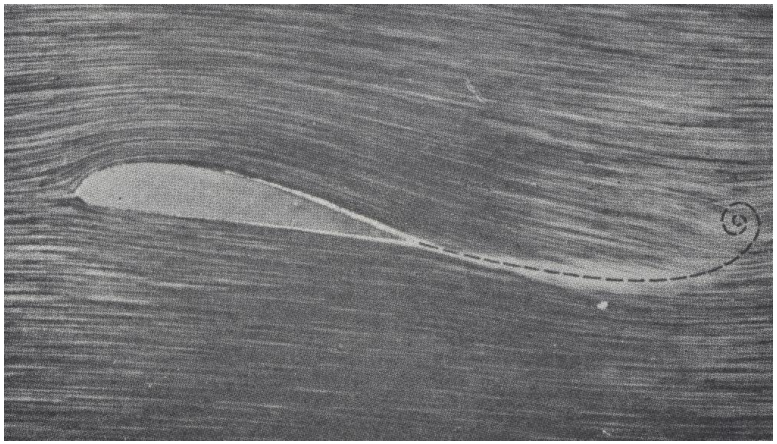
The Kutta Condition





Small Γ

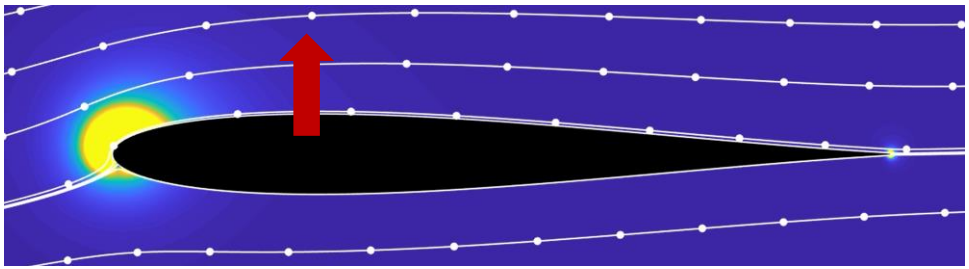




Large Γ



Kutta-Zhukovsky Condition (1902, 1906, 1910)!

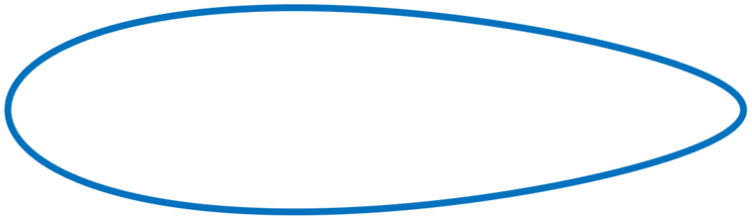




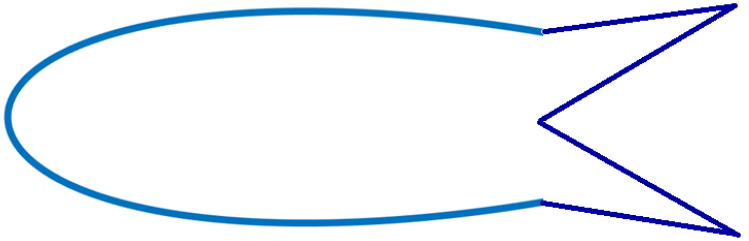
Circulation and lift are such that there is a smooth flow off the sharp trailing edge.

The Current Fragile Theory of Lift

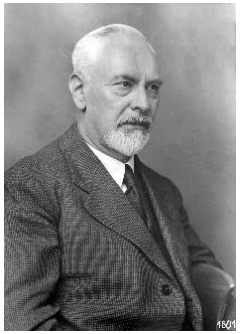
What if there are no sharp edges?!

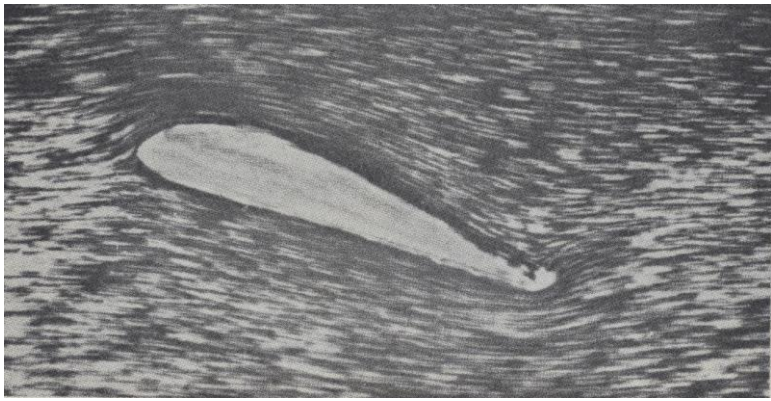


What if multiple sharp edges?!



What about Unsteady Flows?!

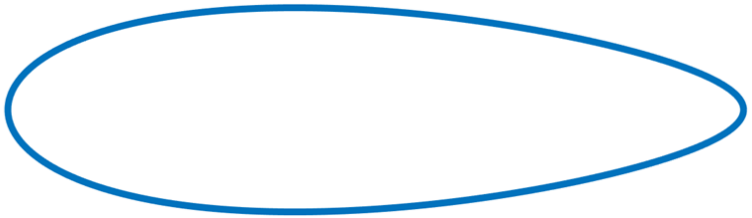




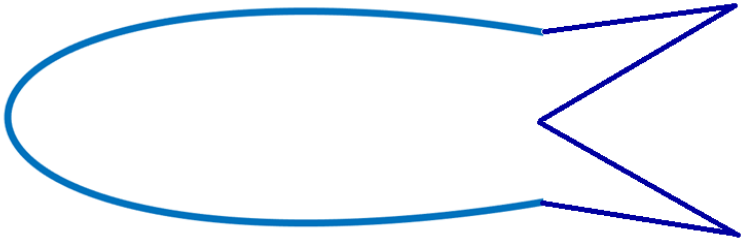
Woolston (1951), Rott and George (1955), Abramson and Chu (1958, 1959, 1967), Henry (1961), Chu (1962), Shen and Crimi (1965), Orszag and Crow (1970), Basu and Hancock (1978), Daniels (1978), Satyanarayana and Davis (1978), Bass et al. (1982), Ansari et al. (2006), Pitt Ford and Babinsky (2013), Hemati et al. (2014), Xia and Mohseni (2017), Taha and Rezaei (2019), Zhu et al. (2020).

The Current Fragile Theory of Lift

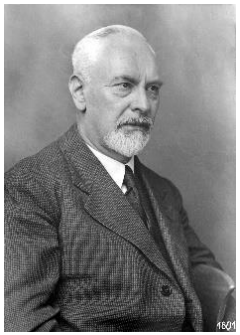
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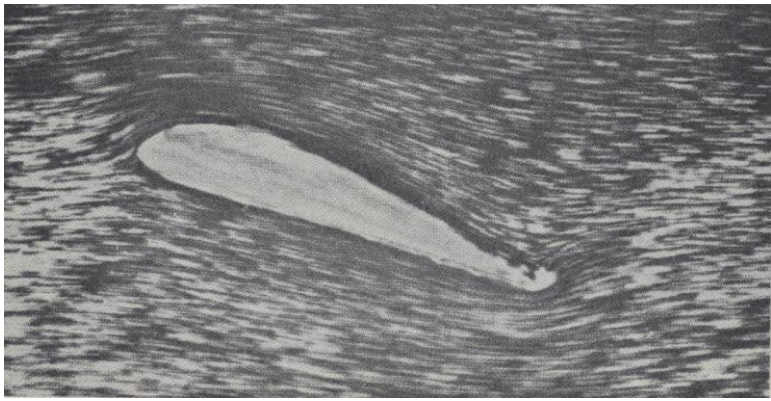


What if multiple sharp edges?!



What about Unsteady Flows?!





Wagner (1925), Theodorsen (1935), Von Karman and Sears (1937), …, Baddoo et al. (JFM 2021)

There are no alternatives!

Indeterminacy of Euler ("Lift is a viscous phenomenon")

Euler

$$\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p$$

Lift is indeterminate!

Navier-Stokes

$$\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p + \upsilon \boldsymbol{\nabla}^2 \boldsymbol{u}$$

Lift is determinate!

Euler + Kutta → Lift

The Kutta condition is a viscous condition!

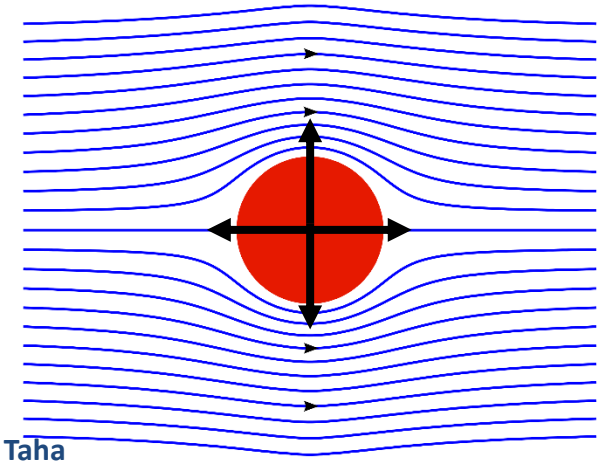
Anderson (6th ed., 4.5.1): *"Nature enforces the Kutta condition by means of friction."*

Ackroyd (2015): "*We fly, therefore, courtesy of the air's very small viscosity*".

Only One Solution!

D'Alembert Paradox (1752)





Zero Force!

"The Kutta Condition is a Viscous Condition"

THE
PHYSICAL REVIEW
A journal of experimental and theoretical physics established by E. L. Nichols in 1893
SECOND SERIES, VOL. 108, NO. 5 DECEMBER 1, 1957

Observation of Perfect Potential Flow in Superfluid*

PAUL P. CRAIG† AND JOHN R. PELLAM
Norman Bridge Laboratory of Physics, California Institute of Technology, Pasadena, California
(Received August 9, 1957)

Only Symmetric Shapes!

Paul Craig Dissertation Caltech 1959:

The many helpful and interesting discussions with Professor R. Feynman of the Physics Department and Professor P. Lagerstrom, Professor H. Liepmann, and Professor A. Roshko of the Aeronautics Department and their continued interest in the investigation are gratefully acknowledged.

undergo pure potential flow without dissipation. The classical viscosity boundary condition at the trailing edge (Kutta condition) does not apply.

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Lack of Dynamics (Indeterminacy) in Potential Flow

Flow Dynamics

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p + \upsilon \boldsymbol{\nabla}^2 \boldsymbol{u} \qquad \text{Dynamics}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \qquad \text{Kinematics}$$

Potential Flow

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \quad \text{and} \quad \boldsymbol{u} = \boldsymbol{\nabla} \phi \;\Rightarrow\; \boldsymbol{\nabla}^2 \phi = 0$$

Potential Flow is Purely Kinematical!
No Dynamics!

A true closure/auxiliary condition must come from Dynamics$\equiv F = ma$

Gonzalez & Taha, *JFM* (2022)

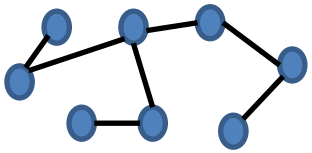
Variational Formulation is the Panacea

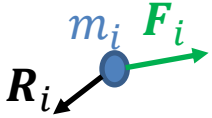
Dynamics

Newtonian ($F = ma$)

Variational/Lagrangian (Least Action)

100 particles
Constraints → 3 DOF: q_1, q_2, q_3





$$m_i \boldsymbol{a}_i = \boldsymbol{F}_i + \boldsymbol{R}_i, i = 1, \dots, 100$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla u} = -\boldsymbol{\nabla} p$$

≡

Nature minimizes $\mathcal{A}$

$$\mathcal{A} = \mathcal{A}(q_1, q_2, q_3)$$

$$\frac{\partial \mathcal{A}}{\partial q_j} = 0, j = 1,2,3$$

- Classical Theory: $\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{x}; \Gamma)$
- $\min_{\Gamma} \mathcal{A}(\Gamma) \rightarrow \frac{\partial \mathcal{A}}{\partial \Gamma} = 0$
- $\min_{\Gamma} \mathcal{A}(\Gamma) = \int \boldsymbol{u}^2 d\Omega$ → $\Gamma^* = 0$: No Lift!



What is the magic $\mathcal{A}$ that Nature minimizes in every airfoil problem?!

A Variational Theory of Lift

Building Intuition/Conjecture

- A true closure/auxiliary condition should come from Dynamics.
- Variational principles would be particularly useful.

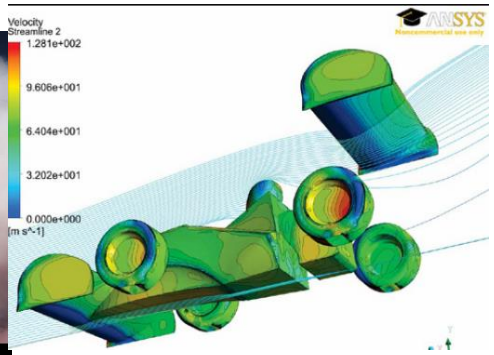
What does Nature minimize in every airfoil problem?!

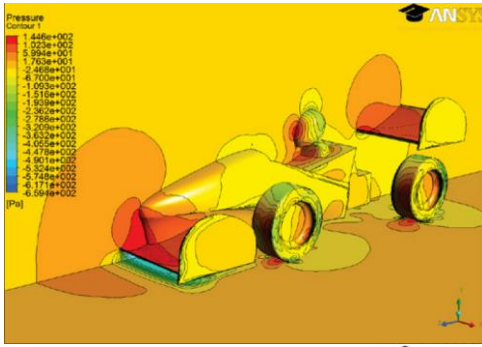


Cody Gonzalez

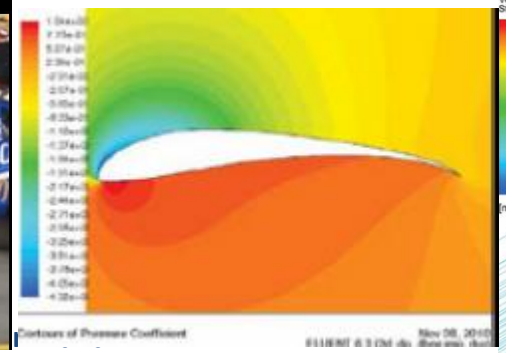
Sanity Check for RANS: large curvature/normal accelerations → X

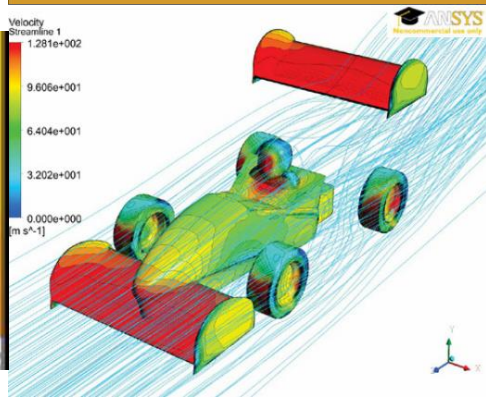










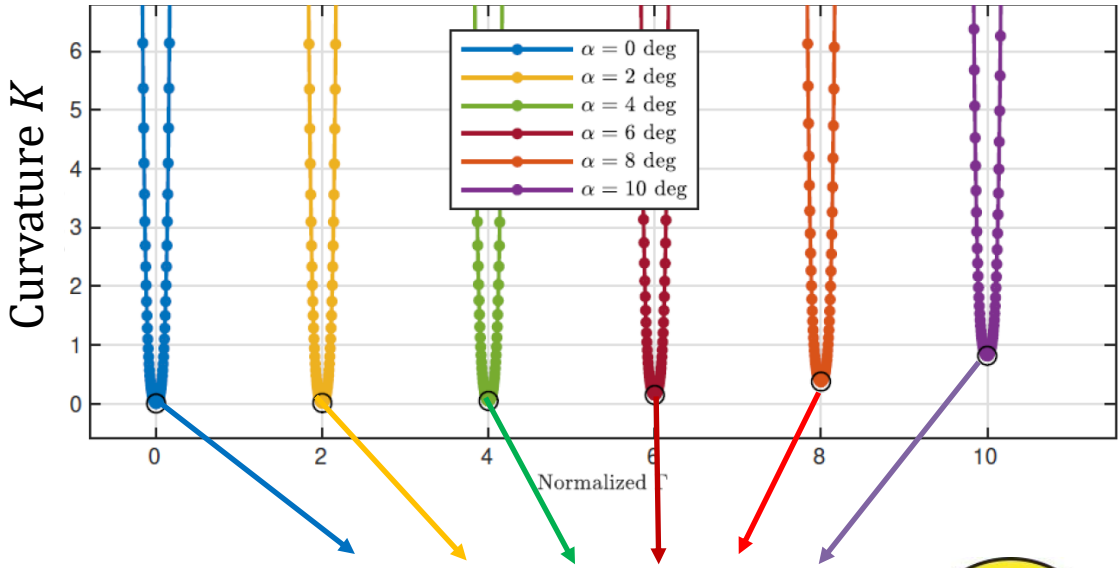


A Variational Theory of Lift

Kutta = Minimum Curvature

- Classical Theory: $\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{x};\Gamma)$
- Curvature: $K(\Gamma)=\frac{1}{2}\int \boldsymbol{a}_n{}^2(\boldsymbol{x};\Gamma)d\Omega$
- $\min_{\Gamma} K(\Gamma)$





Kutta's Values !



Theoretical Basis?!



Gauss' Principle of Least Constraint!

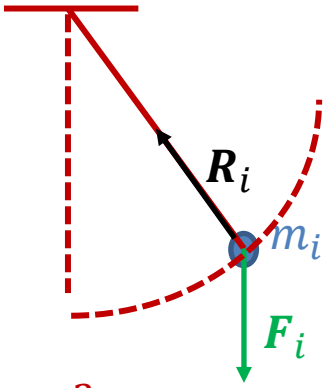
Gauss' Principle of Least Constraint

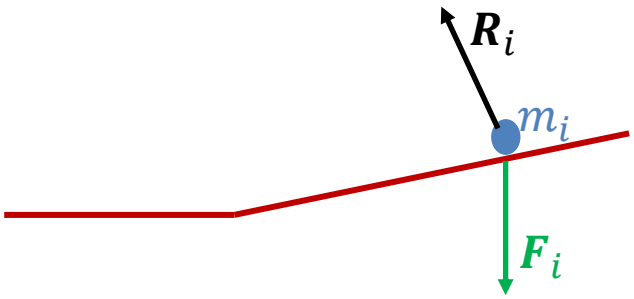
On a New General Fundamental Principle of Mechanics (1829)



Jacobi (1848):

$m_i \boldsymbol{a}_i = \boldsymbol{F}_i + \boldsymbol{R}_i$





$Z = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\boldsymbol{a}_i - \frac{\boldsymbol{F}_i}{m_i}\right)^2$ is minimum

$Z = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\frac{\boldsymbol{R}_i}{m_i}\right)^2$ Least Constraint

"It is quite remarkable that Nature modifies free motions incompatible with the necessary constraints in the same way in which the calculating mathematician uses least squares".

Papastavridis: "*In most of the 20th century English literature, GP has been barely tolerated as a clever but essentially useless academic curiosity, when it was mentioned at all*".

Udwadia and Kalaba (1992)

Gerling
Plücker
Klein
Lindemann
Hilbert
Schmidt
Bochner
Lagerstrom
Van Dyke
Nayfeh
Taha

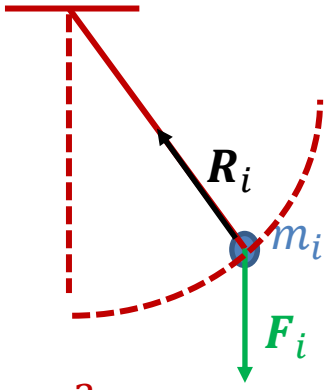
Hertz' Principle of Least Curvature

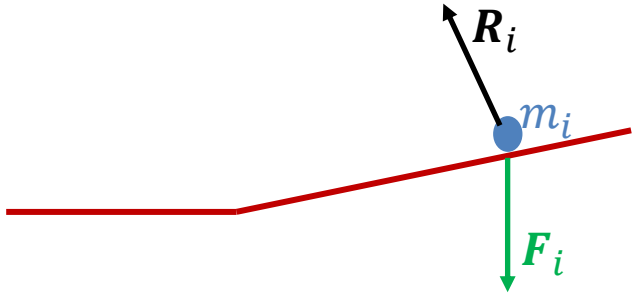
On a New General Fundamental Principle of Mechanics (1829)



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$Z = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\boldsymbol{a}_i - \frac{\boldsymbol{F}_i}{m_i}\right)^2$ is minimum

If no applied/impressed forces $\boldsymbol{F}_i = 0$ $\longrightarrow$ $m_i \boldsymbol{a}_i = \boldsymbol{R}_i$

$\mathcal{A} = \frac{1}{2}\sum_{i=1}^{N} m_i \boldsymbol{a}_i{}^2 = S$ The ***Appellian*** $\longrightarrow$ $\mathcal{A} = \frac{1}{2}\sum_{i=1}^{N} m_i \boldsymbol{a}_{n_i}{}^2$ is minimum

Hertz' Principle of Least Curvature!

Application to Fluid Mechanics

Pressure Forces are Constraint Forces

$$\boldsymbol{a} = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla u} = \underline{-\boldsymbol{\nabla} p} + \underline{\boldsymbol{\nabla} \cdot \boldsymbol{\tau}}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \text{in } \Omega$$
$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \text{on } \partial\Omega$$

$\int \boldsymbol{\nabla} p \cdot \boldsymbol{u} d\Omega = 0$ Pressure forces are workless through divergence-free fields!

Helmholz-Hodge Decomposition: $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\nabla} f, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$

The total magnitude of the pressure gradient over the domain is minimum at every instant!

The Principle of Minimum Pressure Gradient!

Taha & Gonzalez arXiv:2112.12261v4

The Principle of Minimum Pressure Gradient (PMPG)

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla u} = \underline{-\boldsymbol{\nabla} p} + \underline{\boldsymbol{\nabla} \cdot \boldsymbol{\tau}}$$

Theorem (Taha and Gonzalez 2022):

If $\frac{\partial \boldsymbol{u}}{\partial t}\ (\boldsymbol{x}; t)$ minimizes the functional

$$\mathcal{A} = \frac{1}{2} \int \left| \underbrace{\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla u} - \boldsymbol{\nabla} \cdot \boldsymbol{\tau}}_{\boldsymbol{\nabla} p} \right|^2 d\Omega \,, \forall\, t \in \mathcal{R}$$

subject to the constraint $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$

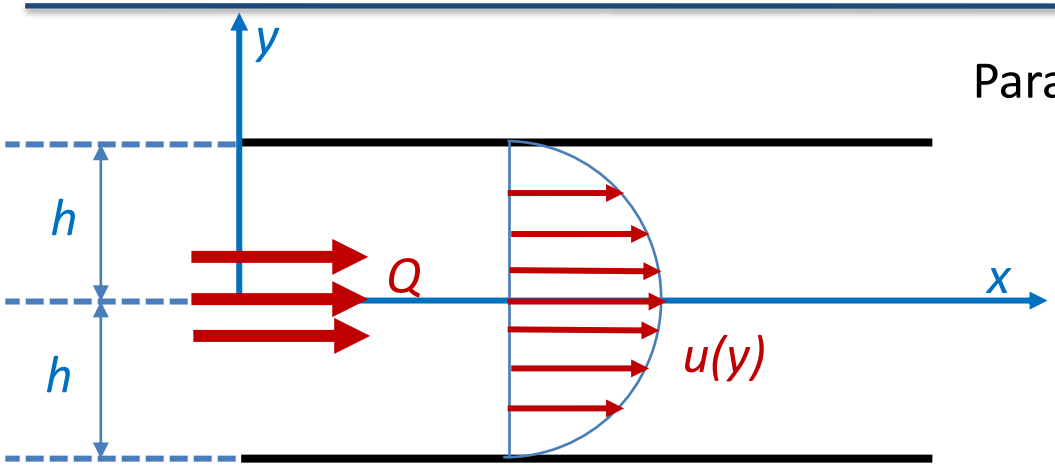
and the no-penetration boundary condition $\boldsymbol{u} \cdot \boldsymbol{n} = \sqrt{}$

Then, $\frac{\partial \boldsymbol{u}}{\partial t}\ (\boldsymbol{x}; t)$ must satisfy

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla u} = -\boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot \boldsymbol{\tau}, \qquad \forall\, \boldsymbol{x} \in \Omega, t \in \mathcal{R}$$

Fluid Mechanics As a Minimization Problem!

Channel Flow



Parameterization: $\boldsymbol{u} = (u(y), 0)$

$$\mathcal{A}(u) = \frac{1}{2}\int_{-h}^{h}\left(\frac{\partial^2 u}{\partial y^2}\right)^2 dy$$

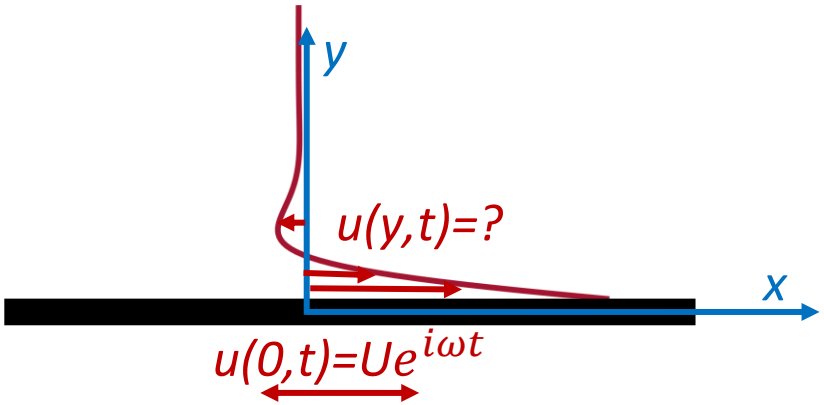
$$\int_{-h}^{h} u dy = Q$$

Find $u(y)$ that minimizes $\mathcal{A}(u)$ subject to the constraint $\int_{-h}^{h} u dy = Q$ and the boundary conditions $u(-h) = u(h) = 0$.

Euler's Isoperimetric Problem !

Fluid Mechanics As a Minimization Problem!

Stokes' Second Problem



Parameterization: $\boldsymbol{u} = (\phi(t)\psi(y), 0)$

$$\mathcal{A} = \frac{1}{2}\int_0^\infty \left(\psi\dot{\phi} - v\psi''\phi\right)^2 dy$$

Temporal: $\mathcal{A} = \frac{1}{2}\left(c_2\dot{\phi}^2 + 2c_1\dot{\phi}\phi + c_1\phi^2\right), c_2 \neq 0$

$$\frac{\partial\mathcal{A}}{\partial\dot{\phi}} = 0 \rightarrow \dot{\phi} = \frac{c_1}{c_2}\phi$$

Spatial: $\mathcal{A}(\psi) = \frac{1}{2}\phi^2\int_0^\infty \left(\frac{c_1}{c_2}\psi - v\psi''\right)^2 dy$

Find $\psi\,(y)$ that minimizes $\mathcal{A}(\psi)$ subject to the boundary conditions $\psi(0) = 1, \psi(\infty) = 0.$

Taha & Gonzalez arXiv:2112.12261v4

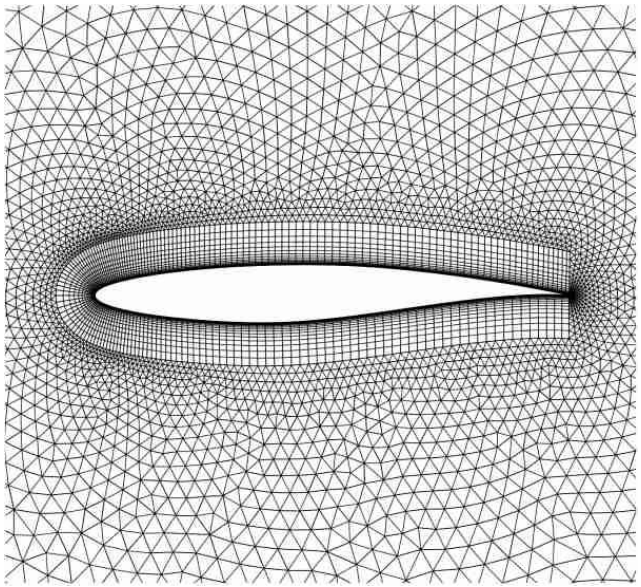
Fluid Mechanics As a Minimization Problem!

Why a minimization principle is better?

Model: $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}; P_1, \dots, P_n)$

Navier-Stokes

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla u} = -\boldsymbol{\nabla} p + \upsilon \boldsymbol{\nabla}^2 \boldsymbol{u}$$



Minimization Principle (PMPG)

$$\mathcal{A} = \frac{1}{2}\int (\boldsymbol{\nabla} p)^2 d\boldsymbol{x} \text{ is minimum}$$

Minimize $\mathcal{A}(P_1, \dots, P_n)$

Flow Physics



Lagrange: *"By the discovery of Euler the whole mechanics of fluids was reduced to a matter of [mathematical] analysis alone."*

Fluid Mechanics As a Minimization Problem!

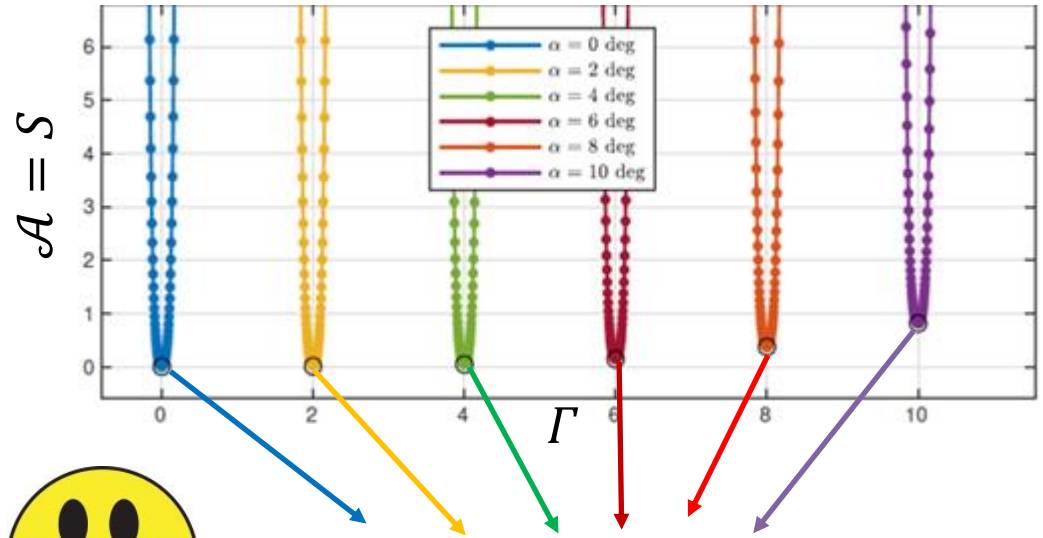
New Lift Theory

Classical Theory: $\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{x};\Gamma)$

$$\mathcal{A} = \int \left[\cancel{\frac{\partial \boldsymbol{u}}{\partial t}} + \boldsymbol{u}\cdot\boldsymbol{\nabla u} - \cancel{\boldsymbol{\nabla}\cdot\boldsymbol{\tau}} \right]^2 d\boldsymbol{x}$$

$$\mathcal{A}(\Gamma) = S(\Gamma) = \int [\boldsymbol{u}(\boldsymbol{x};\Gamma).\boldsymbol{\nabla u}(\boldsymbol{x};\Gamma)]^2 d\boldsymbol{x}$$





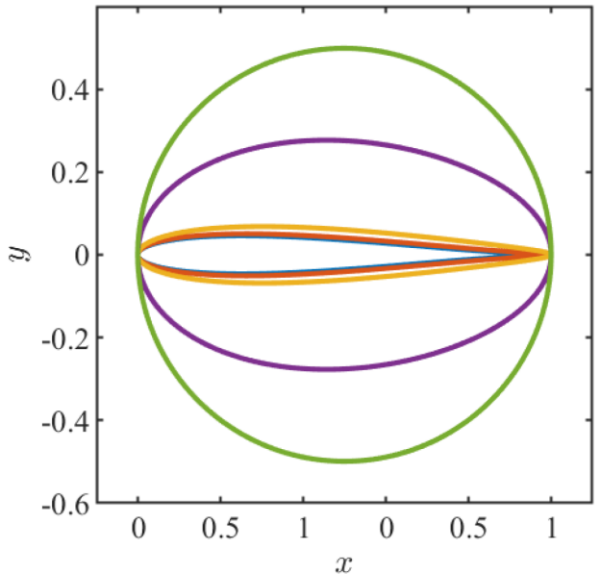


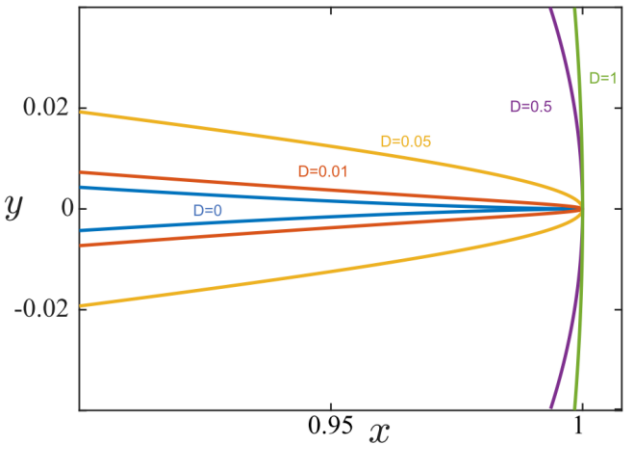
Haithem Taha

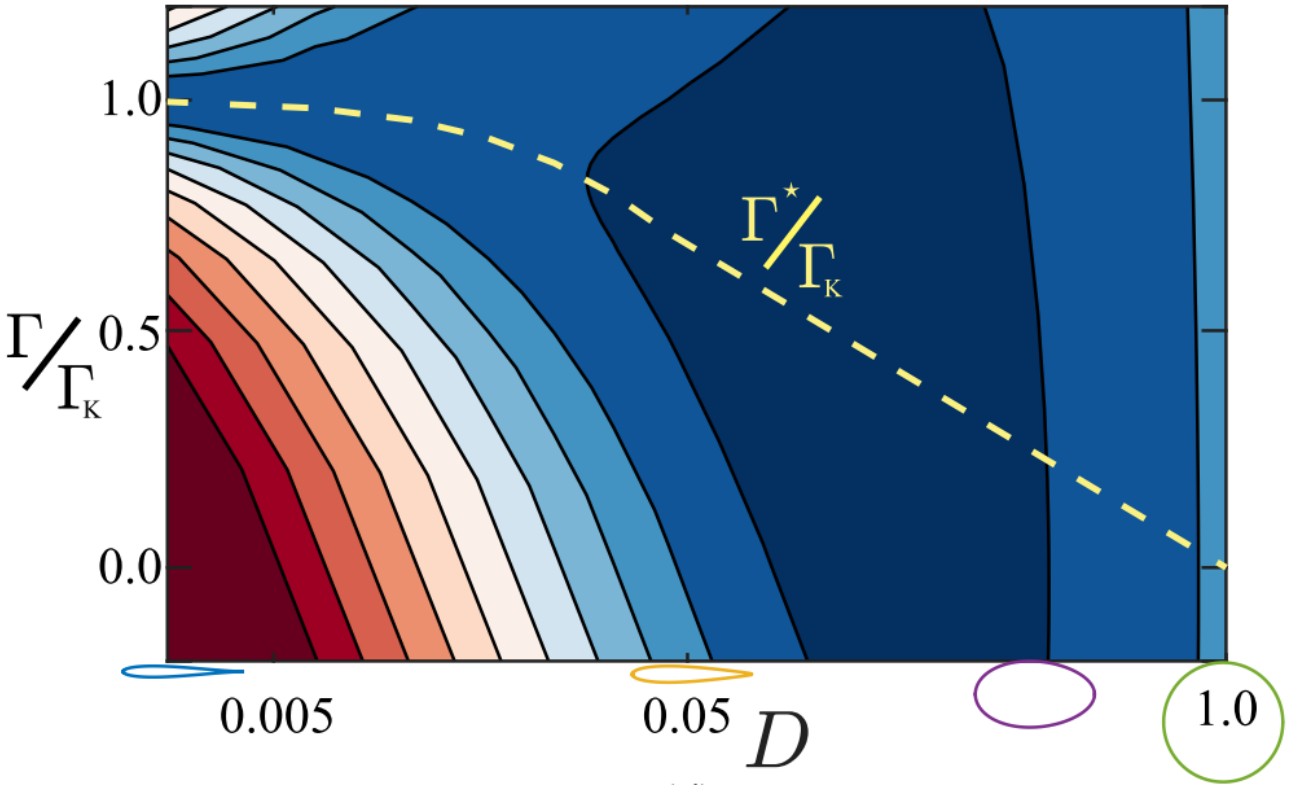
Gonzalez & Taha, *JFM* (2022)

A Variational Theory of Lift

The Kutta Condition is NOT manifestation of Viscous Effects





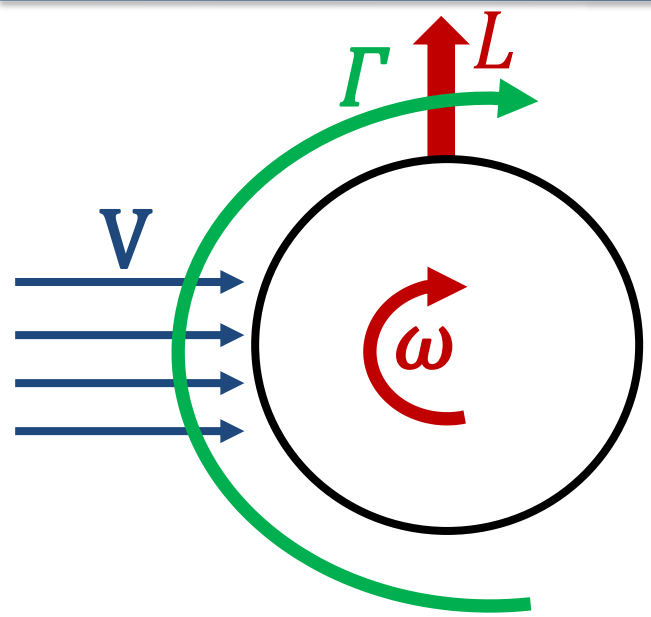


The Kutta condition is NOT a viscous condition!

Gonzalez & Taha, *JFM* (2022)

Fluid Mechanics As a Minimization Problem!

Lift on a Rotating Cylinder





E-ship 1



Mohamed Shorbagy

- Rayleigh (1877):
"friction is the immediate cause of the whirlpool motion".

- Moore (JFM 1957):
"it is not possible to determine Γ without solving the Navier Stokes equations."

Fluid Mechanics As a Minimization Problem!

Lift on a Rotating Cylinder

The flow past a rapidly rotating circular cylinder

By M. B. Glauert

Department of Mathematics, University of Manchester

(*Communicated by M. J. Lighthill, F.R.S.—Received 15 May 1957*)

$$\lambda = \frac{\omega R}{V} > 2 - 3$$

- Prandtl Boundary Layer Equations:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + v\frac{\partial^2 u}{\partial y^2}$$

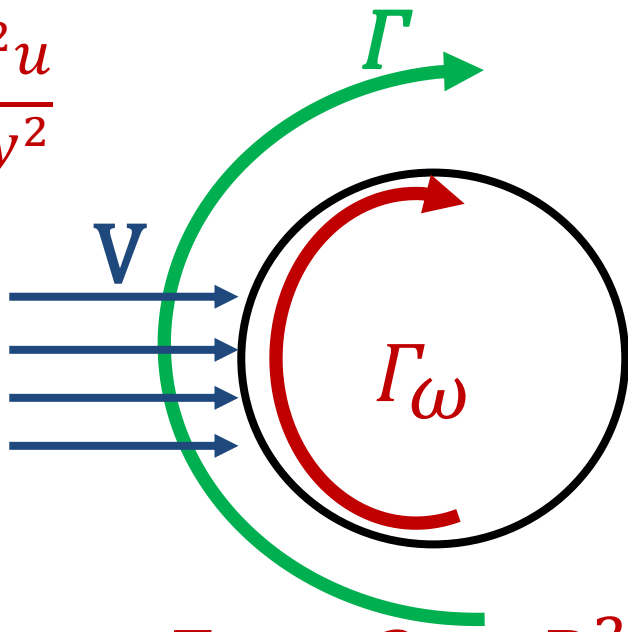
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

- Outer Flow (Potential Flow with Unknown Γ):

$$U = U(x, y; \Gamma)$$

- No Slip Boundary Condition :

$$\frac{\Gamma}{\Gamma_\omega} = 1 - \frac{3}{\lambda^2} - \frac{3.24}{\lambda^4}$$



$$\Gamma_\omega = 2\omega\pi R^2$$

Fluid Mechanics As a Minimization Problem!

Lift on a Rotating Cylinder

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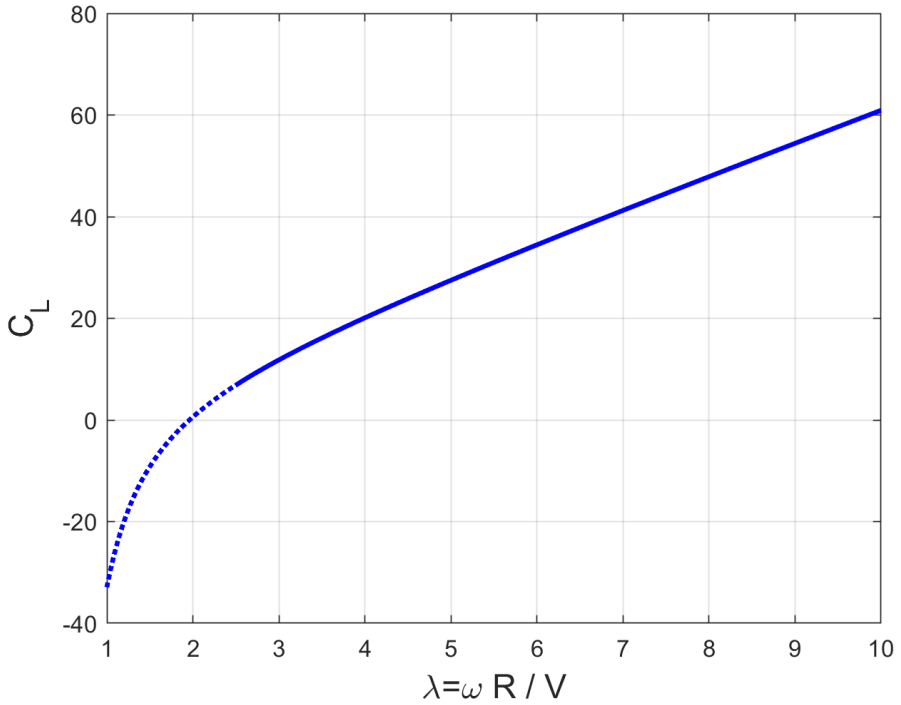
By M. B. Glauert

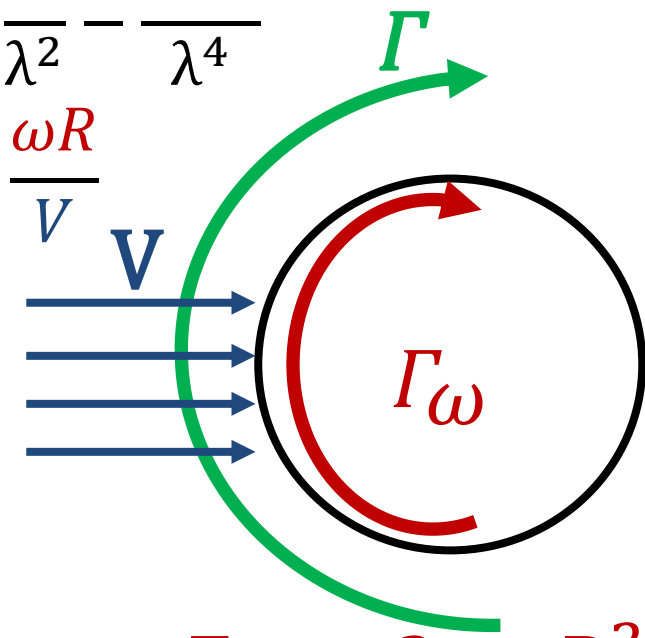
Department of Mathematics, University of Manchester

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$$\frac{\Gamma}{\Gamma_\omega} = 1 - \frac{3}{\lambda^2} - \frac{3.24}{\lambda^4}$$

$$\lambda = \frac{\omega R}{V}$$





$$\Gamma_\omega = 2\omega\pi R^2$$

ha

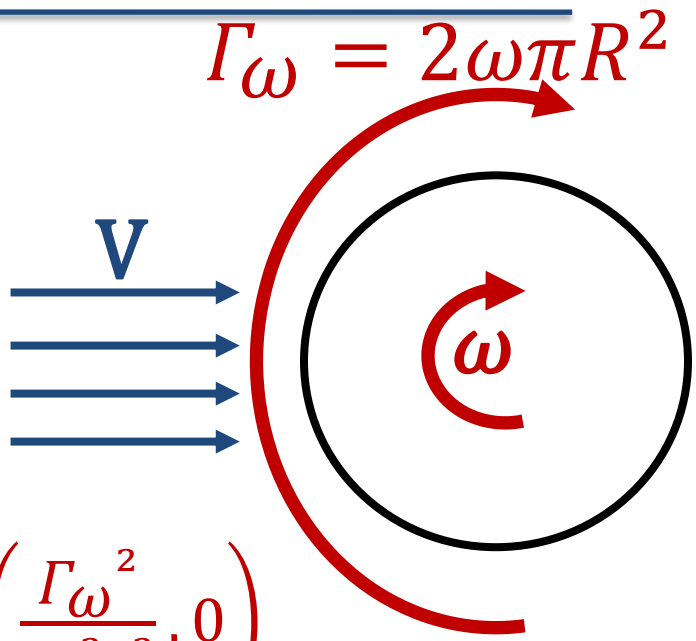
Fluid Mechanics As a Minimization Problem!

Lift on a Rotating Cylinder

Batchelor's "two stages":

1- Rest → $\Gamma_\omega = 2\omega\pi R^2$.

2- Flow over the background flow due to Γ_ω



In Gauss' language:

1- Applied/desired/impressed motion: $\boldsymbol{a}_\omega = \left(\frac{\Gamma_\omega{}^2}{4\pi^2 r^3}, 0\right)$

2- Actual flow will be the closest to the impressed flow (due to Γ_ω)

$$\mathcal{A}(\Gamma) = \int [\boldsymbol{u}(\boldsymbol{x};\Gamma).\boldsymbol{\nabla}\boldsymbol{u}(\boldsymbol{x};\Gamma) - \boldsymbol{a}_\omega]^2 d\boldsymbol{x}$$

$$\frac{\Gamma}{\Gamma_\omega} = \sqrt{1 - \frac{6}{\lambda^2}}$$

Shorbagy & Taha (Under Review)

Fluid Mechanics As a Minimization Problem!

Lift on a Rotating Cylinder

Glauert (Boundary Layer PDE)

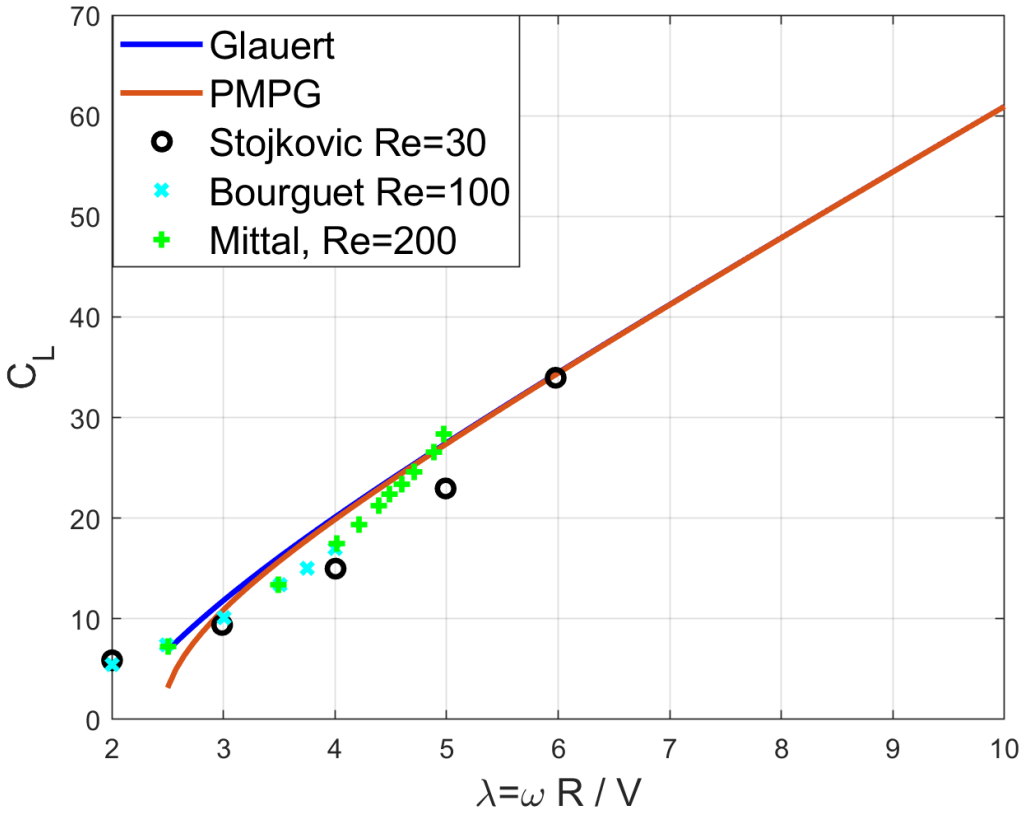
$$\frac{\Gamma}{\Gamma_\omega} = 1 - \frac{3}{\lambda^2} - \frac{3.24}{\lambda^4}$$

Extension → Nonsense

PMPG (1D Minimization)

$$\frac{\Gamma}{\Gamma_\omega} = \sqrt{1 - \frac{6}{\lambda^2}}$$

Extension → $\lambda_{cr} = \sqrt{6} = 2.45$



Shorbagy & Taha (Under Review)

Fluid Mechanics As a Minimization Problem!

Equivalence Between No-Slip & Body Forces

If $\frac{\partial \boldsymbol{u}}{\partial t}\ (\boldsymbol{x}; t)$ minimizes the functional

$$\mathcal{A} = \frac{1}{2}\int \left|\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{a}_{\omega}\right|^{2} d\Omega\,, \forall\ t \in \mathcal{R}$$

subject to the constraint $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$

and the no-penetration boundary condition $\boldsymbol{u}\cdot\boldsymbol{n} = \sqrt{}$

Then, $\frac{\partial \boldsymbol{u}}{\partial t}\ (\boldsymbol{x}; t)$ must satisfy

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = -\boldsymbol{\nabla}p + \boldsymbol{a}_{\omega}, \qquad \forall\ \boldsymbol{x} \in \Omega, t \in \mathcal{R}$$

Equivalence Between No-Slip & Body Forces!

Conclusions

- The magnitude of the pressure gradient over the domain is minimum at every instant!

- The Appellian/curvature is the quantity that Nature minimizes in “every nice” airfoil problem.

- A Variational Theory of Aerodynamics (Generalized Kutta):
  - Derived From First Principles.
  - Applicable for Arbitrary Shape (not confined to airfoils with one sharp edge)
  - The Kutta Condition is NOT a manifestation of viscous effects, but momentum conservation.

Acknowledgment



Cody Gonzalez



Dipan Deb



Mohamed Shorbagy



Jake Singh



Laura Pla Olea



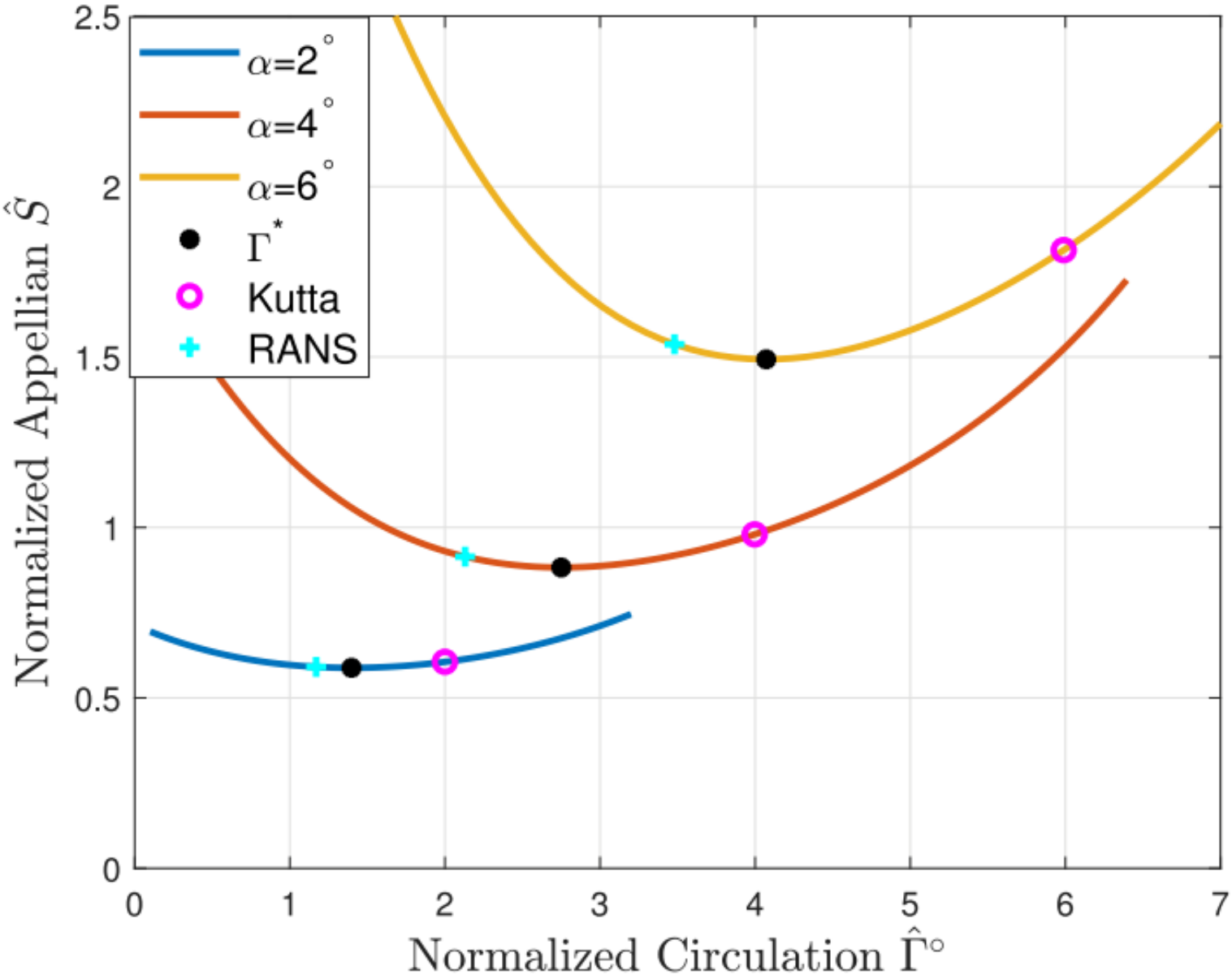
Nabil Khalifa



A Variational Theory of Aero-hydrodynamics

Thank You!

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α=2°
α=4°
α=6°
Γ^*
Kutta
RANS
Normalized Appellian $\hat{S}$
Normalized Circulation $\hat{\Gamma}^\circ$
0
0.5
1
1.5
2
2.5
0
1
2
3
4
5
6
7