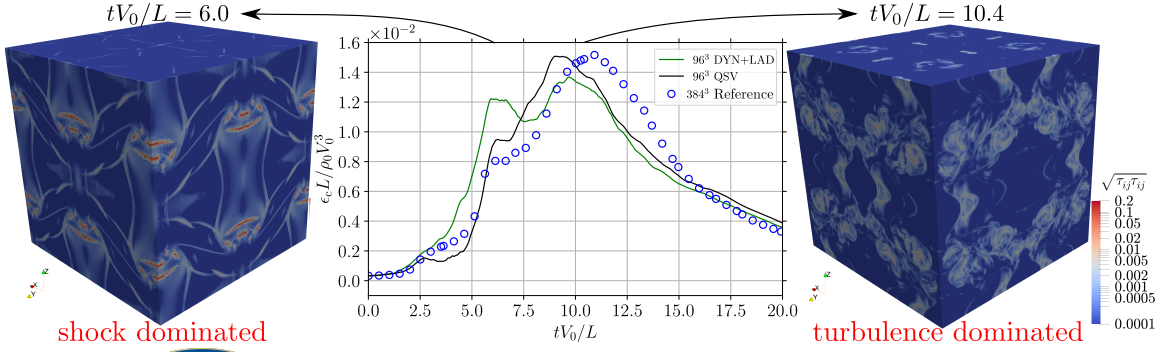
Novel Subfilter Closures For Compressible Flows

Victor Sousa



Office of Naval Research ONR Science & Technology

AFOSR AIR FORCE OFFICE OF SCIENTIFIC RESEARCH

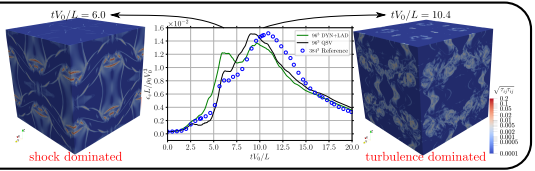
Advanced Modeling & Simulation (AMS)
Seminar Series
NASA Ames Research Center,
July 7th, 2022

PURDUE MECHANICAL ENGINEERING

Outline

Unified shock capturing and LES approach for finite-differences schemes

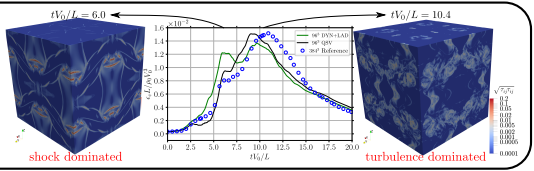
"*A unified Quasi-Spectral Viscosity (QSV) approach to shock capturing and large-eddy simulation*", J. Comp. Phys. (2022)



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Unified shock capturing and LES approach for finite-differences schemes

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Filtered Navier-Stokes as a shock capturing method for flux reconstruction schemes

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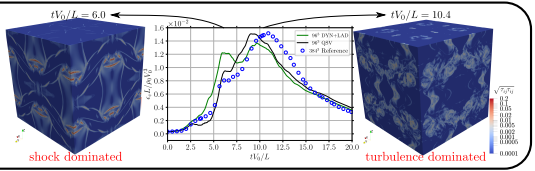


Outline

PURDUE UNIVERSITY

Unified shock capturing and LES approach for finite-differences schemes

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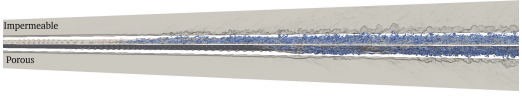
"*A Legendre spectral viscosity (LSV) method applied to shock capturing for high-order flux-reconstruction schemes*", J. Comp. Phys. (2022)



LES of hypersonic BL transition delay

"*Numerical investigation of second-mode attenuation over carbon/carbon porous surfaces*", J. Spacecraft and Rockets (2019)

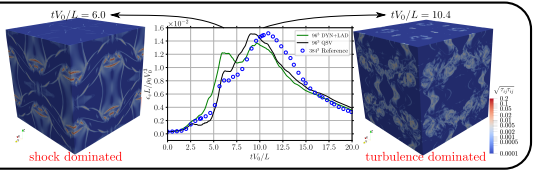
"*Delay of hypersonic transition to turbulence over broadband wall impedance*", to be submitted to J. Fluid Mech.



Outline

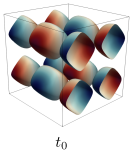
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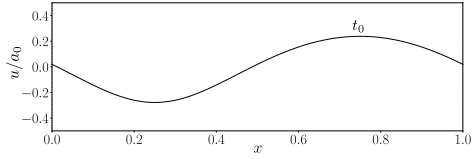
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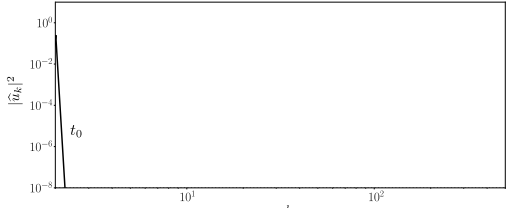


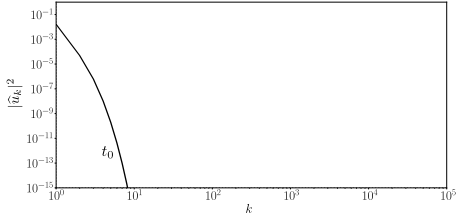
Motivation

Similarities between shocks and turbulence







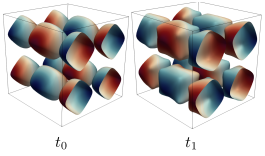


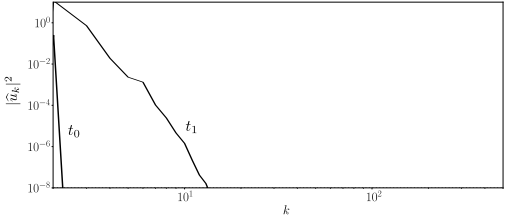
- High energy initialized waves (eddies) steepen (breakdown) transfering energy to smaller length scales, causing thermoviscous (viscous) dissipation to take over.

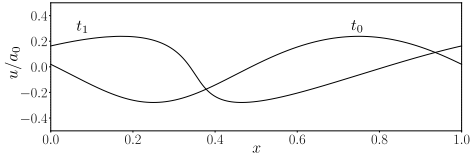
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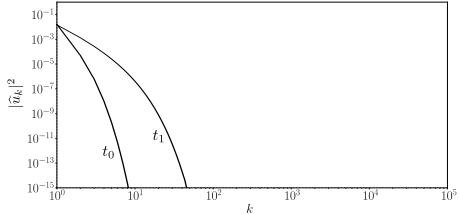
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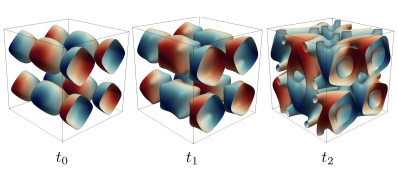


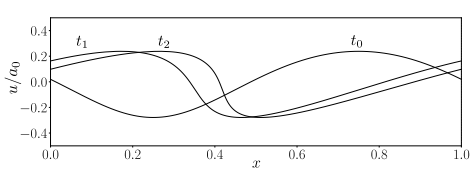
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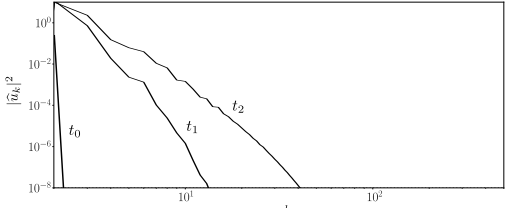
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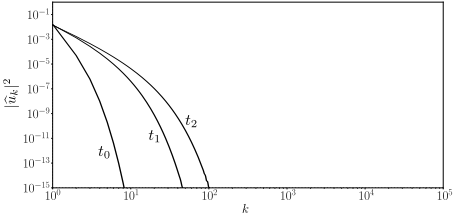
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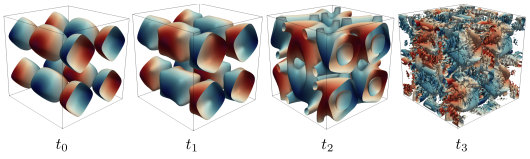


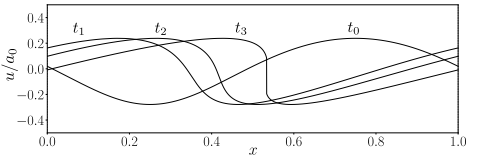
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LES Formalism

- Incompressible Navier-Stokes

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{\partial \mathcal{P}}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

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- Grid filter operation

$$\overline{f}(\mathbf{x}) = \int f(\mathbf{x}')\overline{G}(\mathbf{x}, \mathbf{x}')d\mathbf{x}'$$

LES Formalism

- Grid Filtered Navier-Stokes

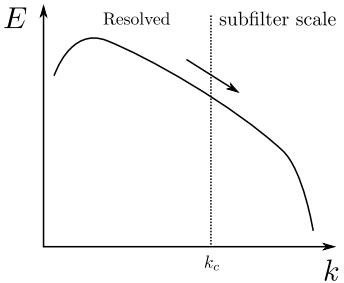
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- $\overline{u_i u_j}$ is unknown

- Grid filter operation

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LES Formalism

- Grid Filtered Navier-Stokes

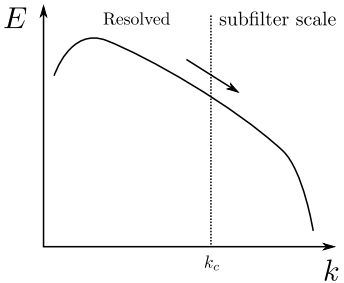
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- Subfilter stress tensor $\tau_{ij} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j$ must be modeled

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$$\overline{f}(\mathbf{x}) = \int f(\mathbf{x}')\overline{G}(\mathbf{x}, \mathbf{x}')d\mathbf{x}'$$



LES Formalism

- Grid Filtered Navier-Stokes

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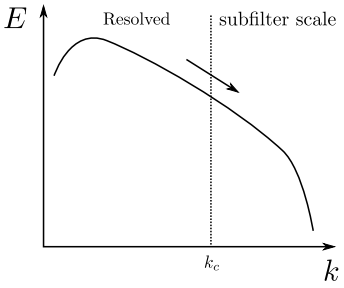
- Subfilter stress tensor $\tau_{ij} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j$ must be modeled
- **Hypothesis:**
  - Energy Transfer Mechanisms $\approx$ Molecular Diffusion

$$-\frac{\partial \tau_{ij}^d}{\partial x_j} = \frac{\partial}{\partial x_j}\nu_t \overline{S}_{ij}$$

$$\tau_{ij}^d = \tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij} \quad \overline{S}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)$$

- Grid filter operation

$$\overline{f}(\mathbf{x}) = \int f(\mathbf{x}')\overline{G}(\mathbf{x}, \mathbf{x}')d\mathbf{x}'$$



- Note:

$$\nu_t = \text{Length Scale}\,[\ell] \times \text{Velocity Scale}\,[\vartheta]$$

LES Formalism

- Grid Filtered Inviscid Burgers' Equation

$$\frac{\partial \overline{u}_1}{\partial t} + \frac{1}{2}\frac{\partial \overline{u}_1 \overline{u}_1}{\partial x_1} = -\frac{1}{2}\frac{\partial \tau_{11}}{\partial x_1}$$

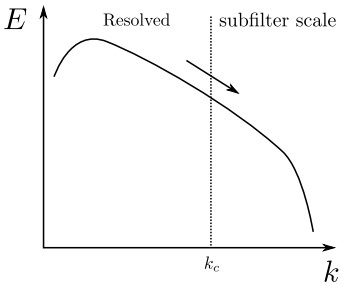
- $\tau_{11} = \overline{u_1 u_1} - \overline{u}_1 \overline{u}_1$ is the subfilter stress tensor and must be modeled
- **Hypothesis:**
  - Energy Transfer Mechanisms $\approx$ Molecular Diffusion

$$-\frac{\partial \tau_{11}}{\partial x_1} = \frac{\partial}{\partial x_1}\nu_t \overline{S}_{11}$$

$$\overline{S}_{11} = \frac{\partial \overline{u}_1}{\partial x_1}$$

- Grid filter operation

$$\overline{f}(\mathbf{x}) = \int f(\mathbf{x}')\overline{G}(\mathbf{x}, \mathbf{x}')d\mathbf{x}'$$



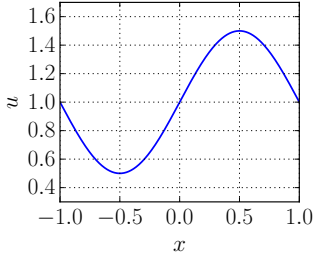
- Note:

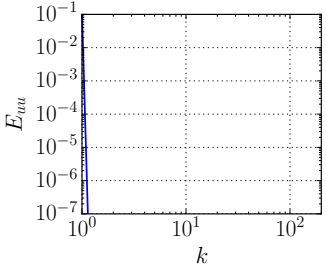
$$\nu_t = \text{Length Scale}\,[\ell] \times \text{Velocity Scale}\,[\vartheta]$$

Filtered Burgers' Equation

- Solve $\frac{\partial \overline{u}_1}{\partial t} + \frac{1}{2}\frac{\partial \overline{u}_1 \overline{u}_1}{\partial x_1} = -\frac{1}{2}\frac{\partial \tau_{11}}{\partial x_1}$ in a periodic domain.
- **What does the exact τ_{11} look like?**

$$\tau_{11} = \overline{u_1 u_1} - \overline{u}_1 \overline{u}_1$$



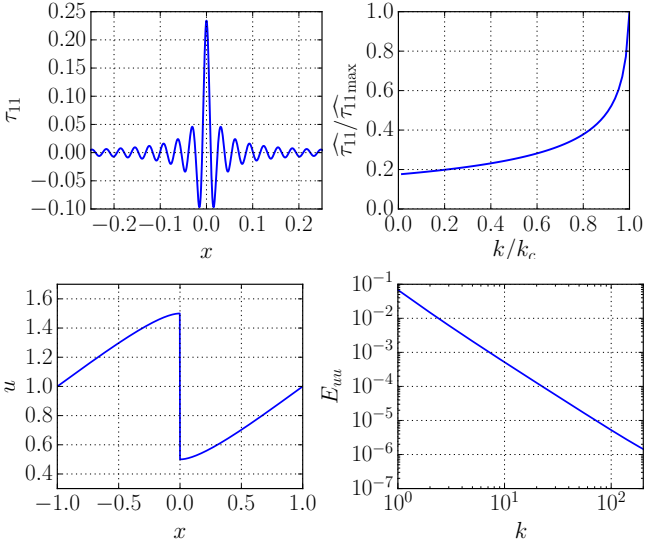


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Filtered Burgers' Equation

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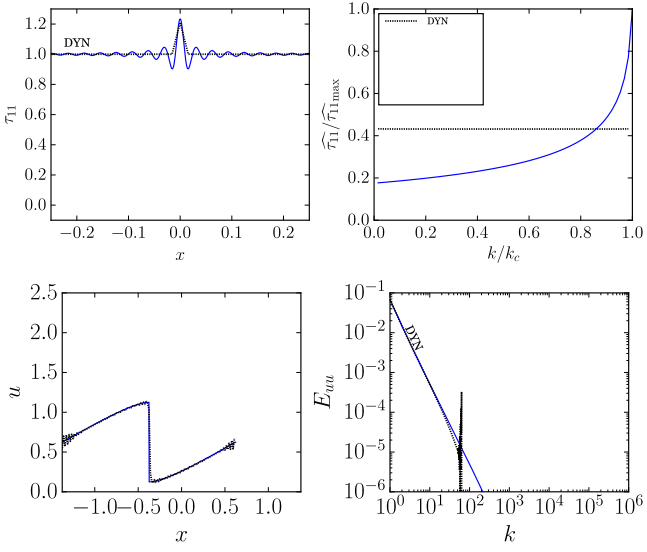


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Filtered Burgers' Equation + Dynamic Smagorinsky (DYN)

- Solve $\frac{\partial \overline{u}_1}{\partial t} + \frac{1}{2}\frac{\partial \overline{u}_1 \overline{u}_1}{\partial x_1} = -\frac{1}{2}\frac{\partial \tau_{11}}{\partial x_1}$ in a periodic domain.

$$-\tau_{11} = \frac{\partial}{\partial x}\left[\nu_t \frac{\partial \overline{u}}{\partial x}\right], \quad \nu_t = C\overline{\Delta}^2 \left|\frac{\partial \overline{u}}{\partial x}\right|.$$

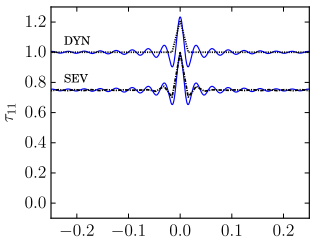


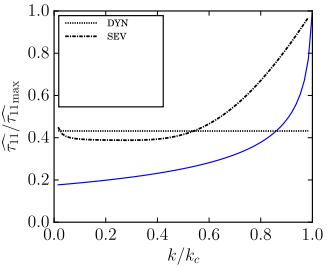
Filtered Burgers' Equation + Spectral Eddy Viscosity (SEV)

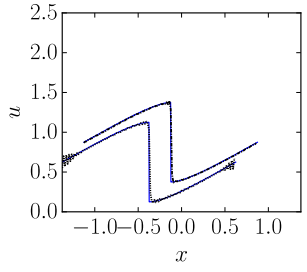
- Solve $\frac{\partial \overline{u}_1}{\partial t} + \frac{1}{2}\frac{\partial \overline{u}_1 \overline{u}_1}{\partial x_1} = -\frac{1}{2}\frac{\partial \tau_{11}}{\partial x_1}$ in a periodic domain.

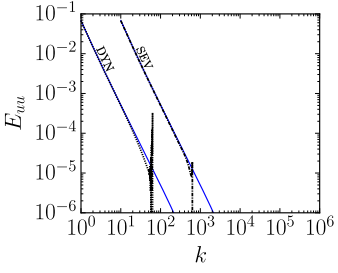
$$-\tau_{11} = \frac{\partial}{\partial x}\left[\sqrt{\frac{E_{k_c}}{k_c}}\mathcal{F}^{-1}[\nu_t^+(k,k_c)] * \frac{\partial \overline{u}}{\partial x}\right],$$

$$\nu_t^+(k/k_c) = 0.267 + 9.21e^{-3.03k_c/k}$$









Unifying Shock and Turbulence Modeling

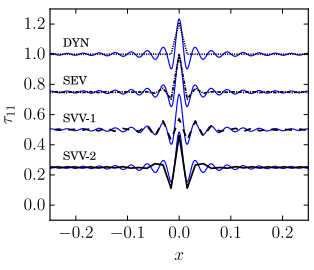
Filtered Burgers' Equation + Spectral Vanishing Viscosity (SVV)

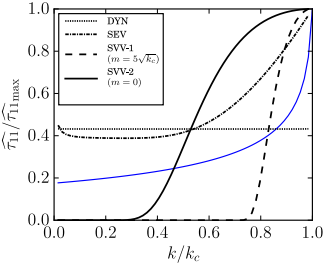
- Solve $\frac{\partial \overline{u}_1}{\partial t} + \frac{1}{2}\frac{\partial \overline{u}_1 \overline{u}_1}{\partial x_1} = -\frac{1}{2}\frac{\partial \tau_{11}}{\partial x_1}$ in a periodic domain.

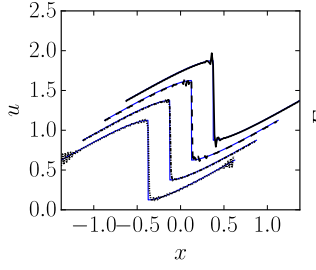
$$-\tau_{11} = \epsilon \frac{\partial}{\partial x}\left[\left(\widehat{Q} * \frac{\partial \overline{u}}{\partial x}\right)\right],$$

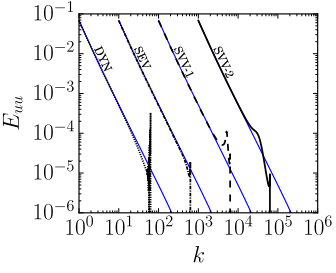
$$\widehat{Q} = \begin{cases} e^{-(k-k_c)^2/(k-m)^2} & \text{if } |k| > m \\ 0, & \text{otherwise} \end{cases}$$

- **SVV-1** $\rightarrow m = 5\sqrt{k_c}$
- **SVV-2** $\rightarrow m = 0$









Unifying Shock and Turbulence Modeling

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Filtered Burgers' Equation + Spectral Vanishing Viscosity (SVV)

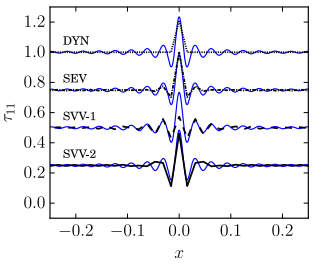
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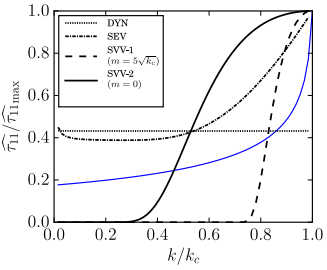
$$-\tau_{11} = \epsilon \frac{\partial}{\partial x}\left[\left(\widehat{Q} * \frac{\partial \overline{u}}{\partial x}\right)\right],$$

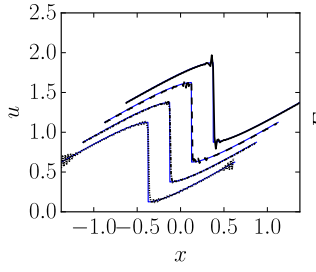
$$\widehat{Q} = \begin{cases} e^{-(k-k_c)^2/(k-m)^2} & \text{if } |k| > m \\ 0, & \text{otherwise} \end{cases}$$

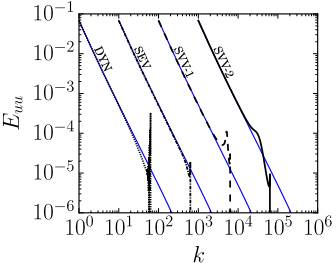
- **SVV-1** $\rightarrow m = 5\sqrt{k_c}$
- **SVV-2** $\rightarrow m = 0$
- Both LES and shock capturing models can be recast in a general format:

$$\tau_{ij} = \ell v_c \left(K * S_{ij}\right).$$







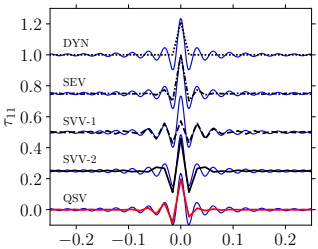


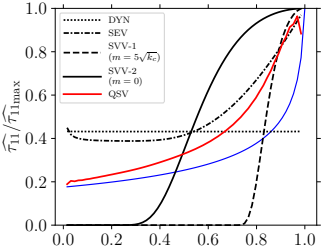
Unifying Shock and Turbulence Modeling

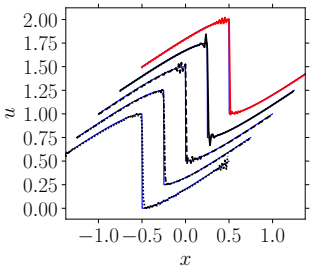
Filtered Burgers' Equation + Quasi-Spectral Viscosity (QSV)

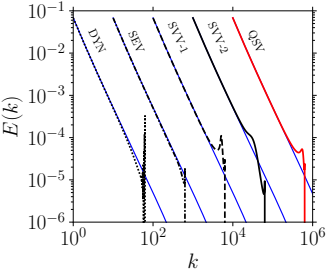
- Solve $\frac{\partial \overline{u}_1}{\partial t} + \frac{1}{2}\frac{\partial \overline{u}_1 \overline{u}_1}{\partial x_1} = -\frac{1}{2}\frac{\partial \tau_{11}}{\partial x_1}$ in a periodic domain.

$$-\tau_{11} = \frac{\partial}{\partial x}\left[\sqrt{2\overline{\Delta} E_{k_c}}(1 - \widetilde{G}_{\text{qsv}}) * \frac{\partial \overline{u}}{\partial x}\right],$$









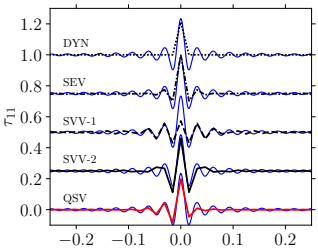
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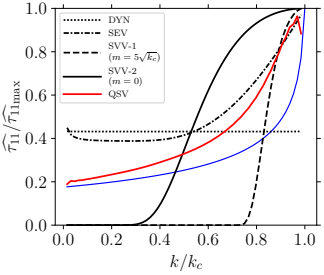
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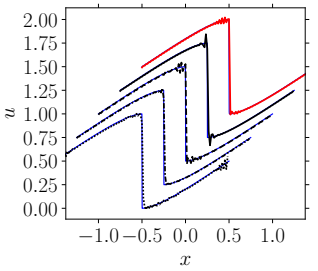
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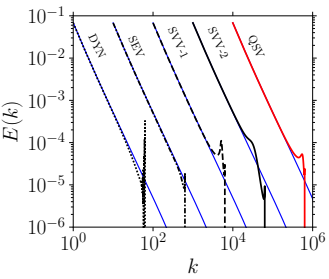
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- **How to find the cutoff energy in physical space?**









Unifying Shock and Turbulence Modeling

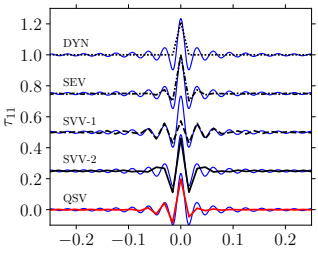
PURDUE UNIVERSITY

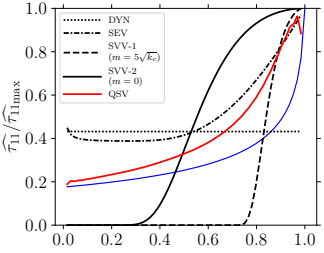
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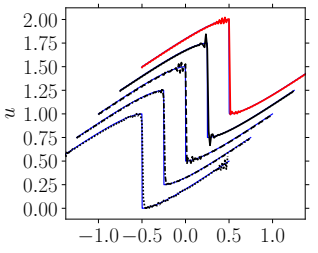
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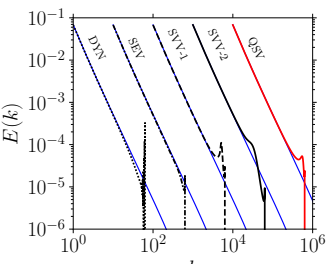
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Unifying Shock and Turbulence Modeling

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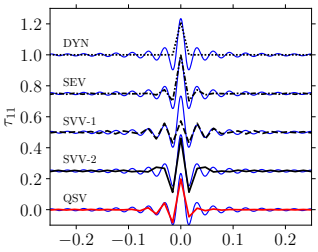
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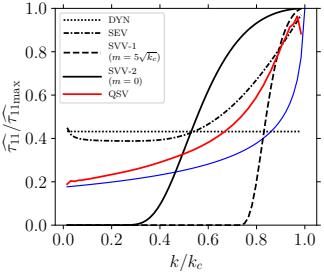
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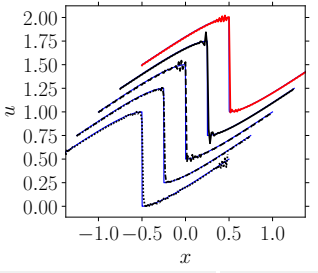
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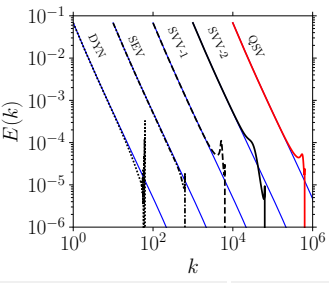
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Low-pass Filter Residuals









Quasi-Spectral Viscosity (QSV)

- **How to find the cutoff energy in physical space?**

Quasi-Spectral Viscosity (QSV)

- **How to find the cutoff energy in physical space?**
- Family of implicit Padé filters:

$$\alpha \tilde{f}_{i-1} + \tilde{f}_i + \alpha \tilde{f}_{i+1} = a_0 f_i + \sum_{j=1}^{3} \frac{a_j}{2}(f_{i+j} + f_{i-j})$$

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$$\widehat{\widetilde{G}}_{\text{Pade}}\left(\alpha, \frac{k}{k_c}\right) = \frac{a_0 + \sum_{j=1}^{3} a_j \cos\left(j\pi\frac{k}{k_c}\right)}{1 + 2\alpha\cos\left(\pi\frac{k}{k_c}\right)}$$

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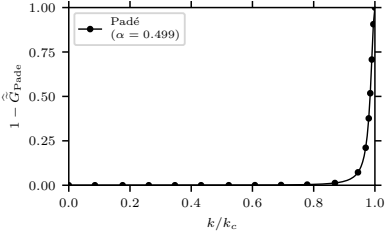
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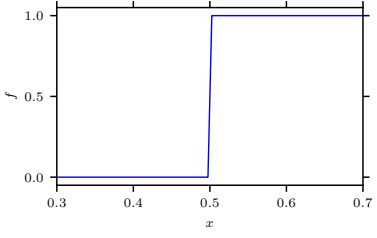
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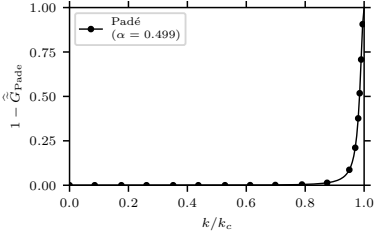
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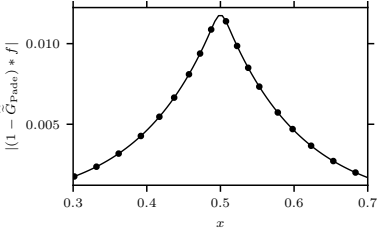
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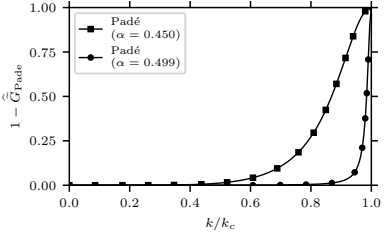
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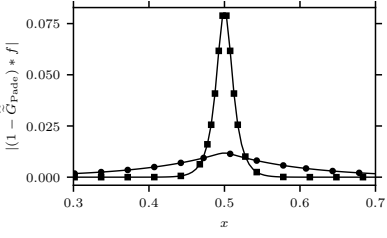
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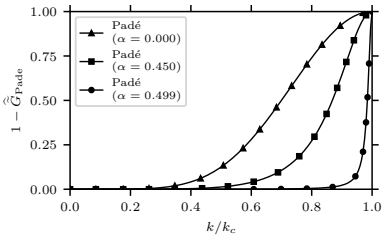
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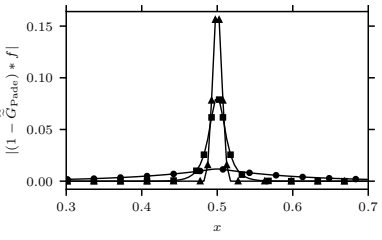
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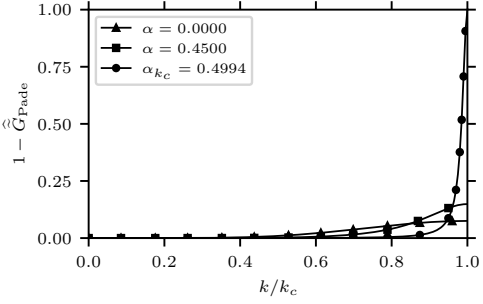
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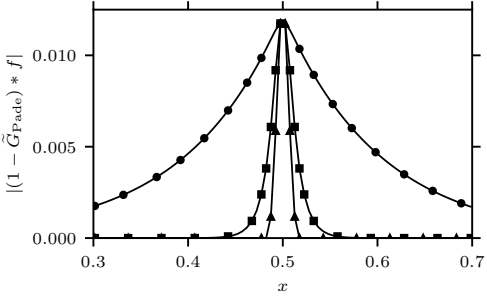
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- Introduce a scaling based on the residual's area:

$$A(\alpha) = \int_0^1 \left(1 - \widehat{\widetilde{G}}_{\text{Pade}}\right) d(k/k_c)$$

- In the end the kinetic energy near the cutoff can be estimated by:

$$E_{k_c} = \left[\left(1 - \widehat{\widetilde{G}}_{\text{Pade}}\right) * \frac{u^2}{2}\right] \frac{A(\alpha)}{A(\alpha_{k_c})}$$





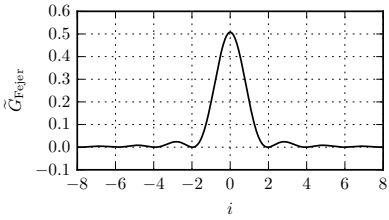
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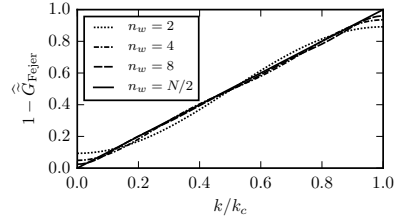
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Quasi-Spectral Viscosity (QSV)

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$$G_{\text{Fejer},ij} = \begin{cases} \frac{4+2M}{2(2+2M)}, & \text{if } i = j \\ \frac{1}{2M(2+2M)} \csc^2\left(\frac{\pi(i-j)}{2M}\right)(1-(-1)^{i-j}), & \text{otherwise.} \end{cases}$$





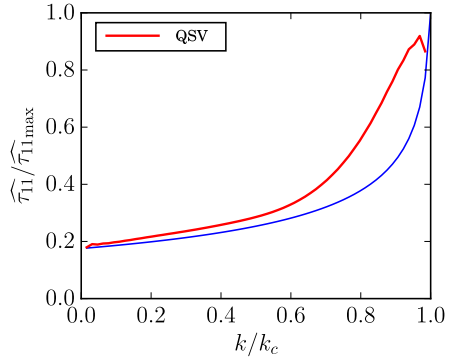
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- Combine it with the Padé filter transfer function to obtain the desired curve:

$$\widehat{\widetilde{G}}_{\text{qsv}} = w\left[1 - \beta\widehat{\widetilde{G}}_{\text{Pade}}(\alpha) + (1-\beta)\widehat{\widetilde{G}}_{\text{Fejer}}\right]$$



Compressible LES Formalism

- Compressible Navier-Stokes

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \mu \sigma_{ij}}{\partial x_j}$$

$$\frac{\partial E}{\partial t} + \frac{\partial (E+p) u_j}{\partial x_j} = -\frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j}\right) + \frac{\partial \mu \sigma_{ij} u_i}{\partial x_j}$$

$$\sigma_{ij} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}$$

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- Grid filter operation

$$\overline{f}(\mathbf{x}) = \int f(\mathbf{x}')\overline{G}(\mathbf{x}, \mathbf{x}')d\mathbf{x}'$$

Compressible LES Formalism

- Grid Filtered Navier-Stokes

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- Nonlinear terms are unknown

- Grid filter operation

$$\overline{f}(\mathbf{x}) = \int f(\mathbf{x}')\overline{G}(\mathbf{x}, \mathbf{x}')d\mathbf{x}'$$

- Favre filter

$$\check{f}(x) = \frac{\overline{\rho f}}{\overline{\rho}}$$

Compressible LES Formalism

- Grid Filtered Navier-Stokes, solving for Favre components

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \overline{\rho}\check{u}_j}{\partial x_j} = 0$$

$$\frac{\partial \overline{\rho}\check{u}_i}{\partial t} + \frac{\partial \overline{\rho}\check{u}_i\check{u}_j}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \mu \check{\sigma}_{ij}}{\partial x_j} - \frac{\partial \overline{\rho}\tau_{ij}}{\partial x_j}$$

$$\frac{\partial \overline{E}}{\partial t} + \frac{\partial (\overline{E}+\overline{p})\check{u}_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(k\frac{\partial \check{T}}{\partial x_j}\right) + \frac{\partial \mu \check{\sigma}_{ij}\check{u}_i}{\partial x_j} - \frac{\partial \overline{\rho} C_p q_j}{\partial x_j}$$

$$\frac{\overline{p}}{\gamma - 1} = \overline{E} - \frac{1}{2}\overline{\rho}\check{u}_i\check{u}_i - \frac{1}{2}\overline{\rho}\tau_{ii}$$

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- Non-negligible terms:
  - $\tau_{ij} = \widetilde{u_i u_j} - \breve{u}_i\breve{u}_j$ is the subfilter stress tensor
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$$\breve{f}(x) = \frac{\overline{\rho f}}{\overline{\rho}}$$

Quasi-Spectral Viscosity (QSV) Closure

$$\tau_{ij} = -C_{\tau_{ij}} \frac{\delta_{i\alpha\beta}\delta_{j\gamma\eta}}{2} \left(\mathcal{D}_{\alpha\gamma} \frac{\partial \ddot{u}_\beta}{\partial x_\eta} + \mathcal{D}_{\gamma\alpha} \frac{\partial \ddot{u}_\eta}{\partial x_\beta} \right)$$

$$q_j = -C_q \delta_{j\alpha\beta} \mathcal{D}_{\alpha\beta} \frac{\partial \ddot{T}}{\partial x_\beta}$$

$$\delta_{i\alpha\beta} = \begin{cases} 1 & \text{if} \quad i = \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Quasi-Spectral Viscosity (QSV) Closure

$$\begin{aligned} \tau_{mn} &= -C_{\tau_{mn}} \frac{1}{2} \left(\mathcal{D}_{mn} \frac{\partial \ddot{u}_m}{\partial x_n} + \mathcal{D}_{nm} \frac{\partial \ddot{u}_n}{\partial x_m} \right) \\ q_n &= -C_q \mathcal{D}_{nn} \frac{\partial \ddot{T}}{\partial x_n} \end{aligned}$$

- No summation implied over the m and n indices

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- No summation implied over the m and n indices

- Dynamic estimation of the dissipation magnitude

$$\mathcal{D}_{mn} = \upsilon_m(\breve{u}_n)\ell_n$$

$$\upsilon_m(\breve{u}_n) = \sqrt{\frac{2E_{k_c}^m(\breve{u}_n)}{\overline{\Delta}_m}} \rightarrow \text{Velocity scale}$$

$$\ell_n = \overline{\Delta}_n \rightarrow \text{Length scale}$$

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- Wavenumber Modulation

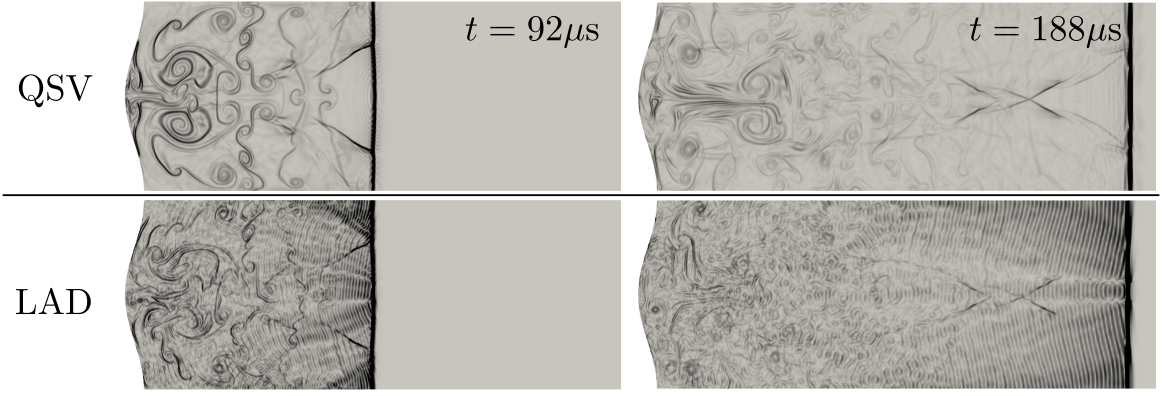
$$\frac{\partial \ddot{u}_m}{\partial x_n} = \frac{\partial \check{u}_m}{\partial x_n} * \left(1 - \tilde{G}_{\text{qsv}}\right)$$

$$\frac{\partial \ddot{T}}{\partial x_n} = \frac{\partial \check{T}}{\partial x_n} * \left(1 - \tilde{G}_{\text{qsv}}\right)$$

Shock/Wavywall Interaction



Shock/Wavywall Interaction



Kawai and Lele, J. Comp. Phys., 2008. Kawai *et al.*, J. Comp. Phys., 2010.

Subsonic Taylor-Green Vortex (TGV)

- Initial fluctuation Mach number

$$M_t = V_0/\sqrt{\gamma p_0/\rho_0} = 0.1$$

- Reynolds number

$$Re = \rho_0 V_0 L/\mu_0 = 5000$$

Turbulence Dominated Flow

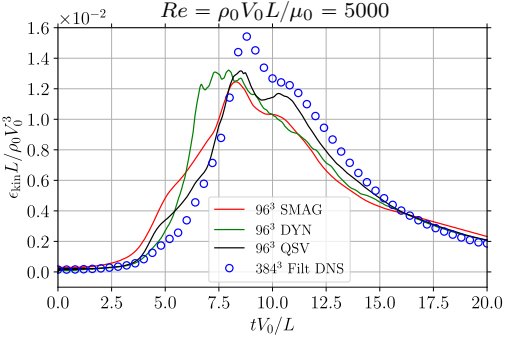
Subsonic Taylor-Green Vortex (TGV)

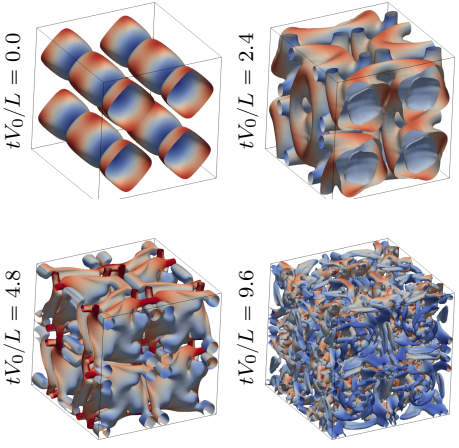
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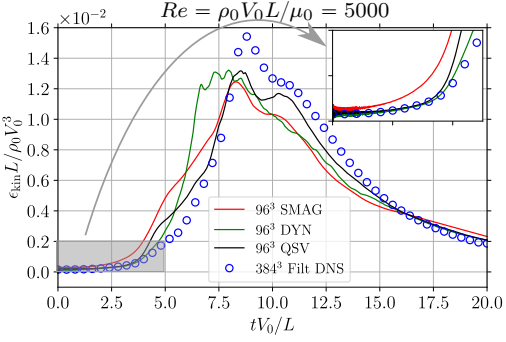
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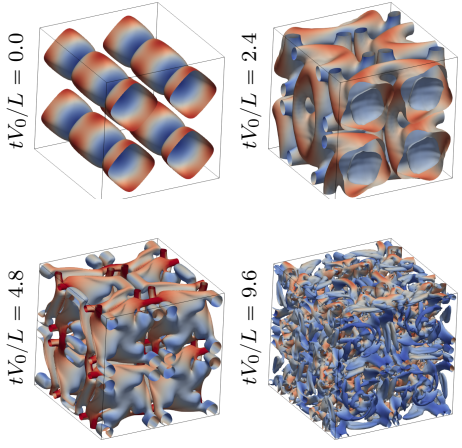
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Supersonic Taylor-Green Vortex (TGV)

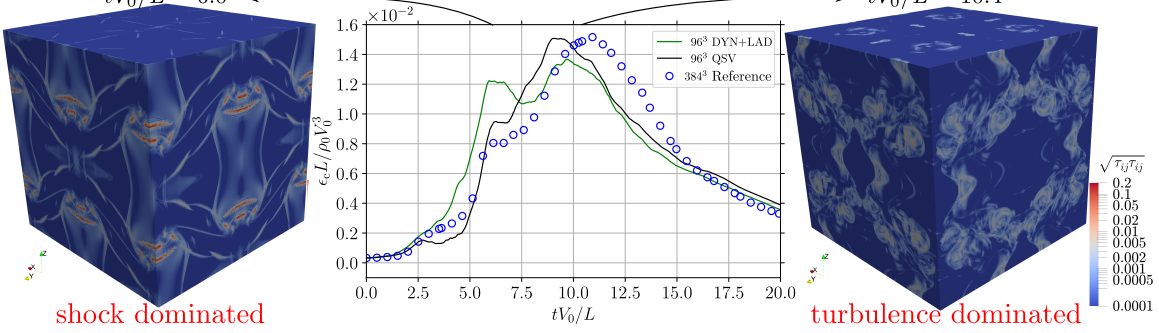
- Initial fluctuation Mach number

$$M_t = V_0/\sqrt{\gamma p_0/\rho_0} = 1.2$$

- Reynolds number

$$Re = \rho_0 V_0 L/\mu_0 = 5000$$

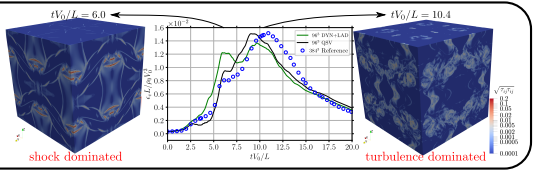
$tV_0/L = 6.0$ ← → $tV_0/L = 10.4$



Outline

Unified shock capturing and LES approach for finite-differences schemes

"*A unified Quasi-Spectral Viscosity (QSV) approach to shock capturing and large-eddy simulation*", J. Comp. Phys. (2022)



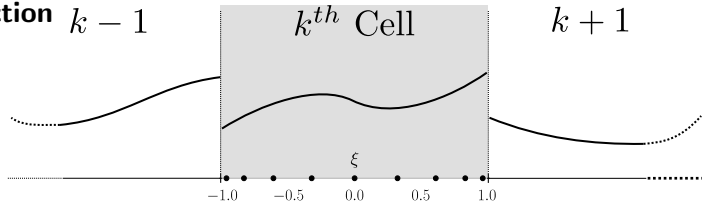
Filtered Navier-Stokes as a shock capturing method for flux reconstruction schemes

"*A Legendre spectral viscosity (LSV) method applied to shock capturing for high-order flux-reconstruction schemes*", J. Comp. Phys. (2022)



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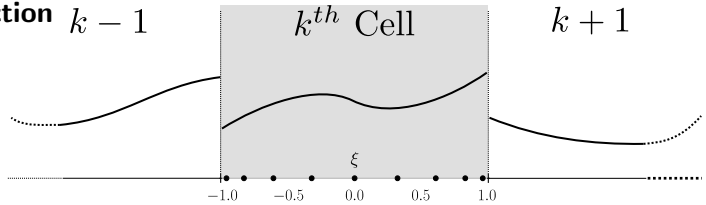
Flux Reconstruction



- Let $\{\xi_j\}_{j=0}^{N}$ the Gauss-Legendre quadrature points.

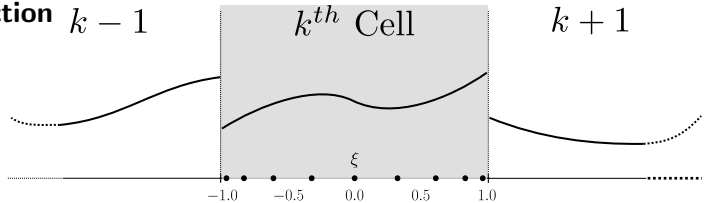
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Flux Reconstruction



- Let $\{\xi_j\}_{j=0}^{N}$ the Gauss-Legendre quadrature points.
- Such points:
  - Decrease the importance of the Runge's phenomenon (Karniadakis and Sherwin, 2013)
  - Minimize aliasing errors in nonlinear flux reconstruction schemes (Jameson et al., 2012)

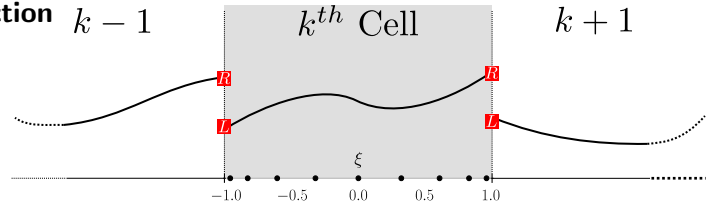
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- Let $\{h_j(\xi)\}_{j=0}^{N}$ be the set of Lagrange polynomials associated with $\{\xi_j\}_{j=0}^{N}$.

$$f_k(\xi) = \sum_{j=0}^{N} f_k(\xi_j) h_j(\xi).$$

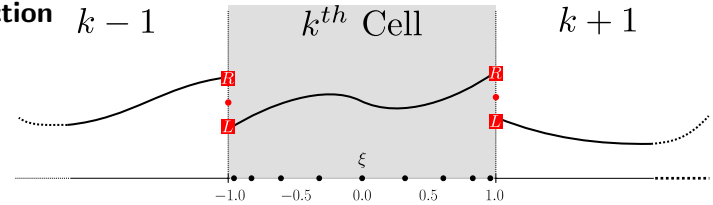
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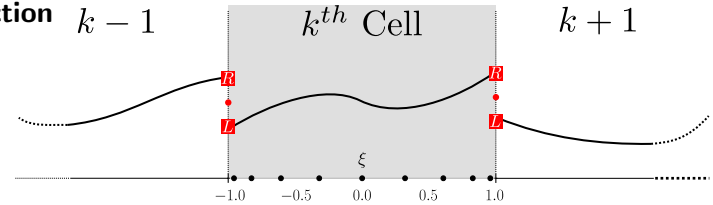
Flux Reconstruction



- A unique flux value at each interface is constructed via a numerical flux algorithm, e.g., Lax-Friedrichs flux:

$$f^{un}_{k-\frac{1}{2}} = \text{Numerical Flux}\left[f_{k-1}(+1), f_k(-1)\right], \quad f^{un}_{k+\frac{1}{2}} = \text{Numerical Flux}\left[f_k(+1), f_{k+1}(-1)\right].$$

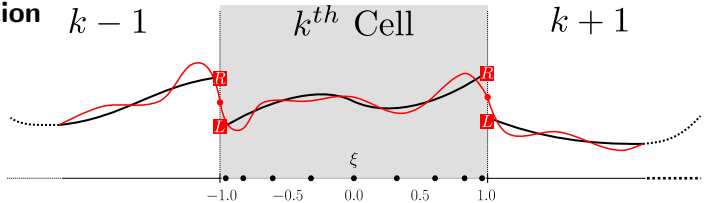
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Flux Reconstruction



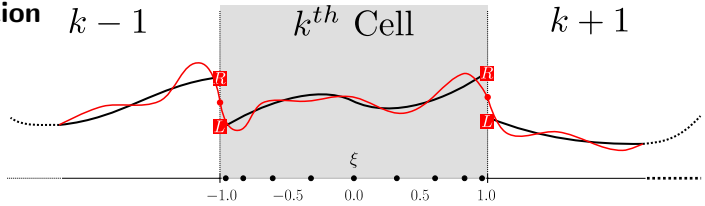
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- Continuous flux function construction

$$f^c_k(\xi) = f_k(\xi) + \left(f^{un}_{k-\frac{1}{2}} - f_k(-1)\right) g_L(\xi) + \left(f^{un}_{k+\frac{1}{2}} - f_k(+1)\right) g_R(\xi).$$

Flux Reconstruction

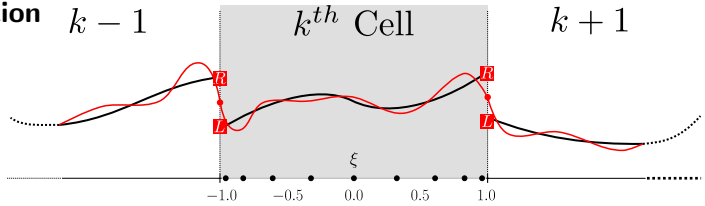


- Divergence of the continuous flux

$$\frac{\partial f_k^c}{\partial \xi}\Big|_{\xi=\xi_i} = D_{ij} f_k(\xi_j) + \left(f^{un}_{k-\frac{1}{2}} - f_k(-1)\right) g'_L(\xi_i) + \left(f^{un}_{k+\frac{1}{2}} - f_k(+1)\right) g'_R(\xi_i).$$

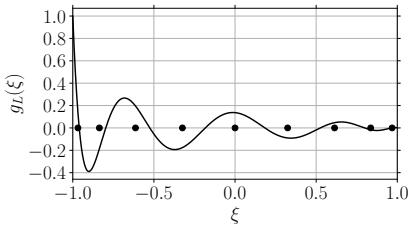
Shock capturing closure of the fNS equations

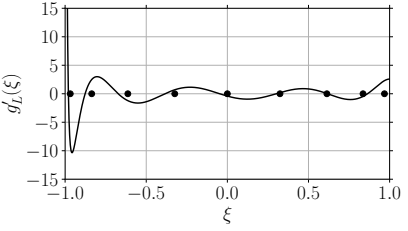
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Shock capturing closure of the fNS equations

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Idea:

- Exploit the relationship between FR schemes and the Legendre orthogonal basis to:
  - Estimate the energy near the cutoff,

$$\tilde{u}_n = \mathcal{L}[\mathbf{u}] = \frac{\langle u, L_n \rangle}{\|L_n\|^2} = \frac{1}{\gamma_n} \sum_{k=0}^{N} u(\xi_k) L_n(\xi_k) w_k,$$

$$E_N^j(\tilde{u}_j) = \left| \frac{1}{\gamma_N} \sum_{k=0}^{N} \frac{1}{2} (\tilde{u}_j(\xi_k^j))^2 L_N(\xi_k^j) w_k \right|;$$

  - Modulate the SFS flux terms,

$$\mathcal{M}[\sigma, \mathbf{u}] = \mathcal{L}^{-1}[\sigma, \mathcal{L}[\mathbf{u}]],$$

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Shock capturing closure of the fNS equations

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Filtered Compressible Navier-Stokes equations

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \overline{\rho}\check{u}_j}{\partial x_j} = 0$$

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$$\frac{\partial \overline{E}}{\partial t} + \frac{\partial (\overline{E}+\overline{p})\check{u}_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(k\frac{\partial \check{T}}{\partial x_j}\right) + \frac{\partial \mu \check{\sigma}_{ij}\check{u}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\frac{1}{2}\left(\frac{\gamma \pi_j}{\gamma - 1} + \overline{\rho} C_p q_j\right)\right]$$

$$\frac{\overline{p}}{\gamma - 1} = \overline{E} - \frac{1}{2}\overline{\rho}\check{u}_i\check{u}_i - \frac{1}{2}\overline{\rho}\tau_{ii}$$

- Sharp spectral Legendre filter operation

$$\overline{f}(\mathbf{x}) = \int f(\mathbf{x}')\overline{G}(\mathbf{x}, \mathbf{x}')d\mathbf{x}'$$

- Favre filter

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Shock capturing closure of the fNS equations

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  - $q_j = \widetilde{T u_j} - \check{T}\check{u}_j$ is the subfilter heat flux
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Legendre Spectral Viscosity (LSV) closure

$$
\begin{aligned}
\tau_{mn} &= -\mathcal{M}\left[\sigma_e, \frac{1}{2}\left(\mathcal{D}_{mn}\frac{\partial \check{u}_m}{\partial \xi_n} + \mathcal{D}_{nm}\frac{\partial \check{u}_n}{\partial \xi_m}\right)\right], \\
q_n &= -\mathcal{M}\left[\sigma_e, \mathcal{D}_{nn}\frac{\partial \check{T}}{\partial \xi_n}\right], \\
\pi_n &= -\mathcal{D}_{nn}\frac{\partial \overline{p}}{\partial \xi_n}.
\end{aligned}
$$

- No summation implied over the m and n indices

Legendre Spectral Viscosity (LSV) closure

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\end{aligned}
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- No summation implied over the m and n indices

- Dynamic estimation of the dissipation magnitude

$$\mathcal{D}_{mn} = \upsilon_m(\check{u}_n)\ell_m$$

$$\upsilon_m(\check{u}_n) = \sqrt{\frac{N}{2}E_N^m(\check{u}_n)} \rightarrow \text{Velocity scale}$$

$$\ell_n = w_n \rightarrow \text{Length scale}$$

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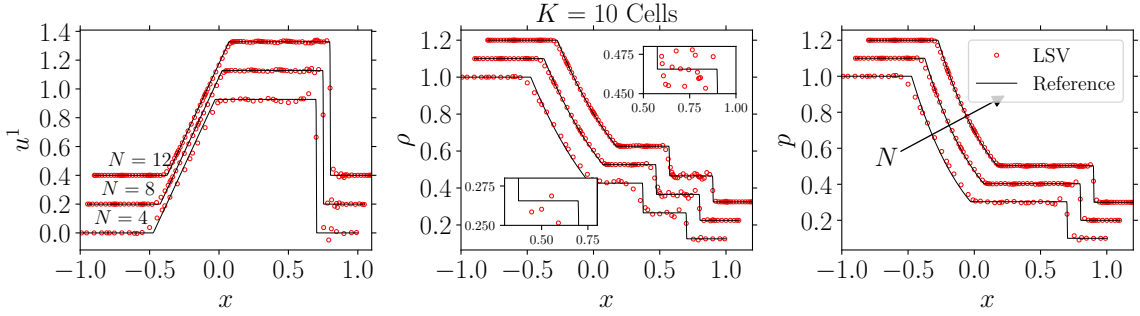
$$\ell_n = w_n \rightarrow \text{Length scale}$$

- Wavenumber Modulation

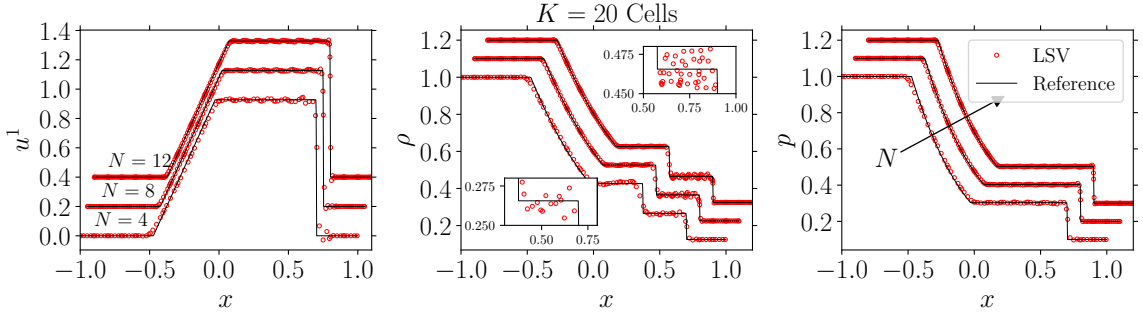
$$\mathcal{M}[\sigma, \mathbf{u}] = \mathcal{L}^{-1}[\sigma, \mathcal{L}[\mathbf{u}]],$$

$$\sigma_e(\eta) = 0.2 + 0.8\frac{e^{4\eta} - 1}{e^4 - 1},$$

Sod shock tube problem

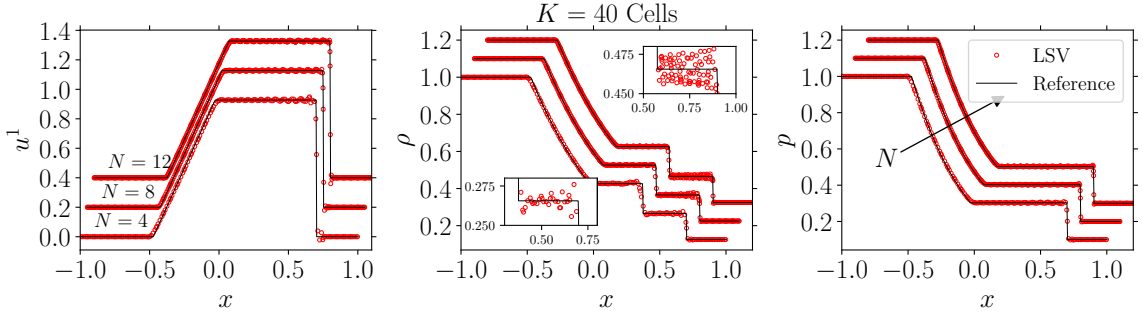


Sod shock tube problem



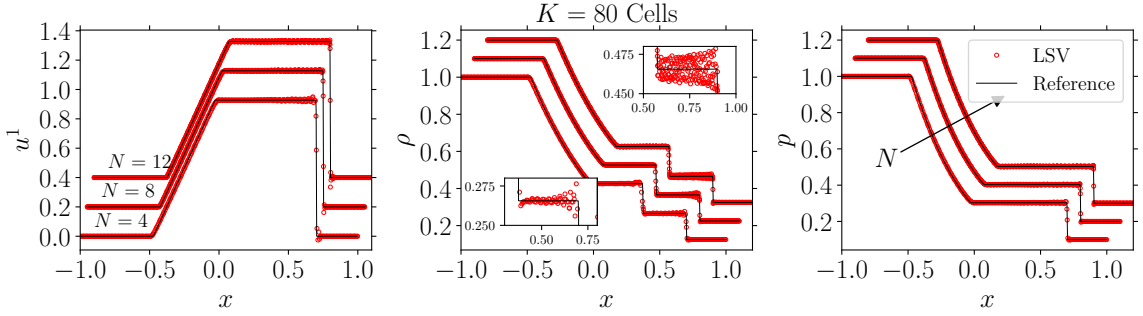
Shock capturing closure of the fNS equations

Sod shock tube problem

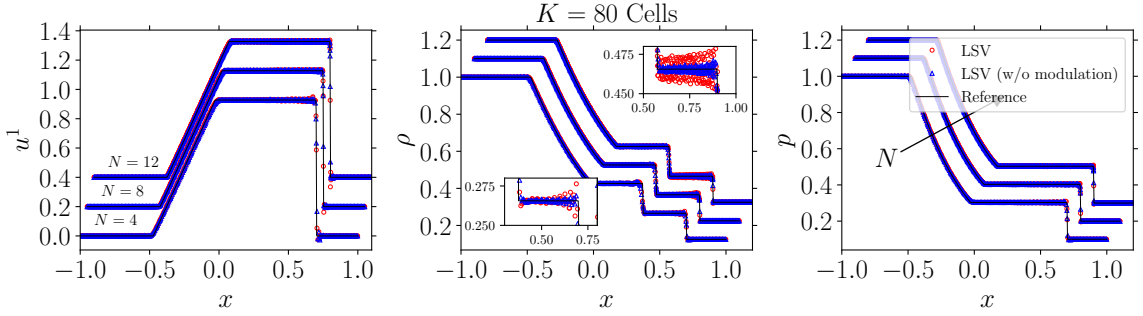


Shock capturing closure of the fNS equations

Sod shock tube problem



Sod shock tube problem



$$\tau_{mn} = -\mathcal{M}\left[\sigma_e, \frac{1}{2}\left(\mathcal{D}_{mn}\frac{\partial \check{u}_m}{\partial \xi_n} + \mathcal{D}_{nm}\frac{\partial \check{u}_n}{\partial \xi_m}\right)\right],$$

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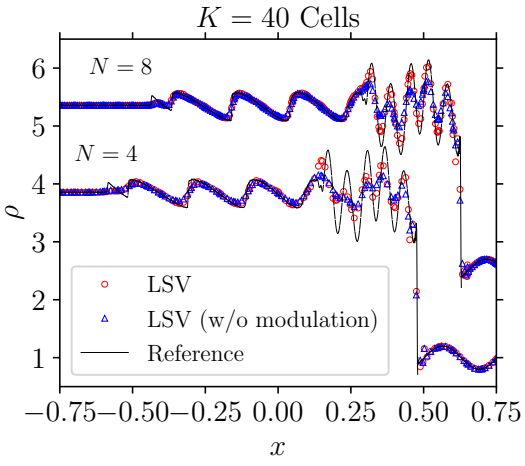
$$\pi_n = -\mathcal{D}_{nn}\frac{\partial \overline{p}}{\partial \xi_n}.$$

$$\tau_{mn} = -\frac{1}{2}\left(\mathcal{D}_{mn}\frac{\partial \check{u}_m}{\partial \xi_n} + \mathcal{D}_{nm}\frac{\partial \check{u}_n}{\partial \xi_m}\right),$$

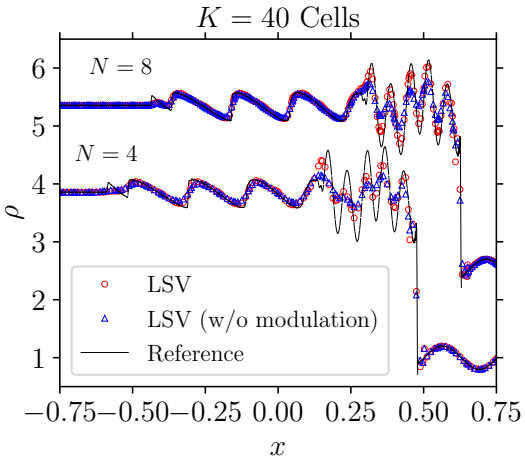
$$q_n = -\mathcal{D}_{nn}\frac{\partial \check{T}}{\partial \xi_n},$$

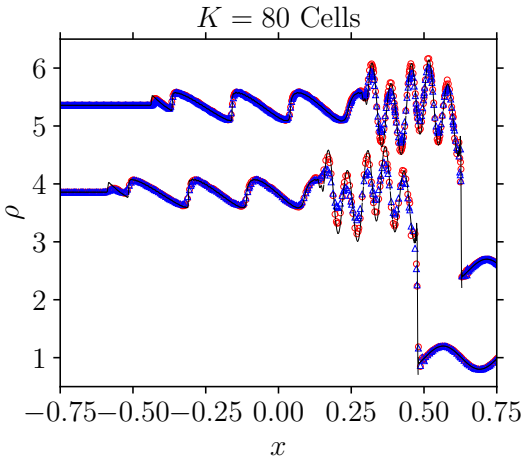
$$\pi_n = -\mathcal{D}_{nn}\frac{\partial \overline{p}}{\partial \xi_n}.$$

Shu-Osher interaction problem



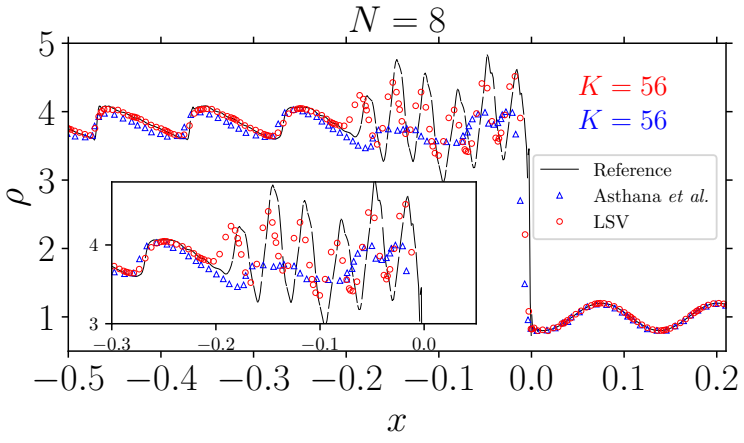
Shu-Osher interaction problem





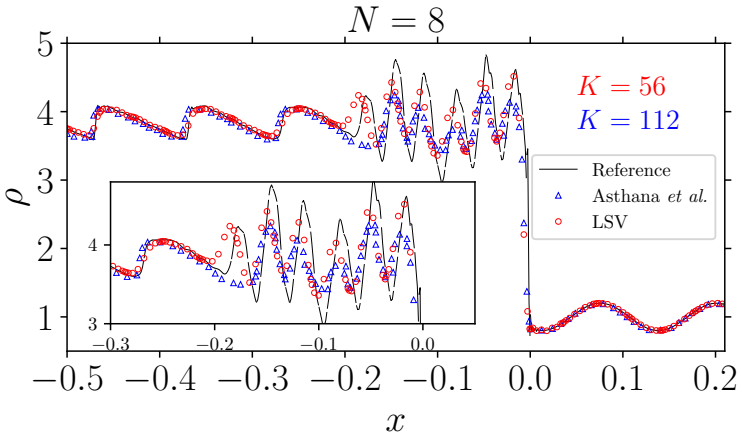
Shock capturing closure of the fNS equations

Shu-Osher interaction problem



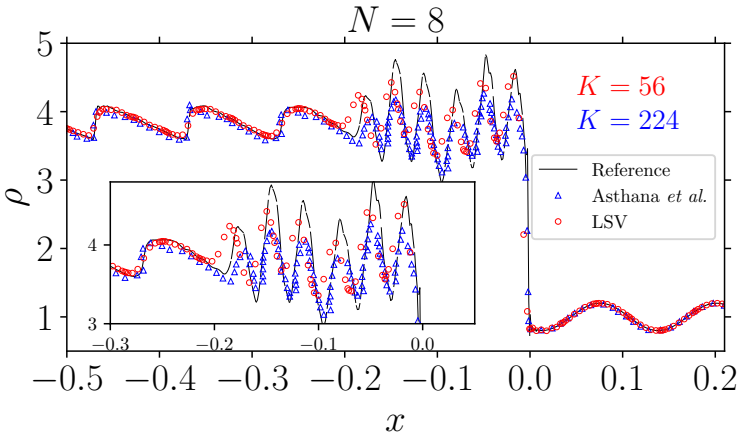
Asthana *et al.*, "Non-linear stabilization of high-order Flux Reconstruction schemes via Fourier-spectral filtering", J. Comp. Phys., 2015.

Shu-Osher interaction problem



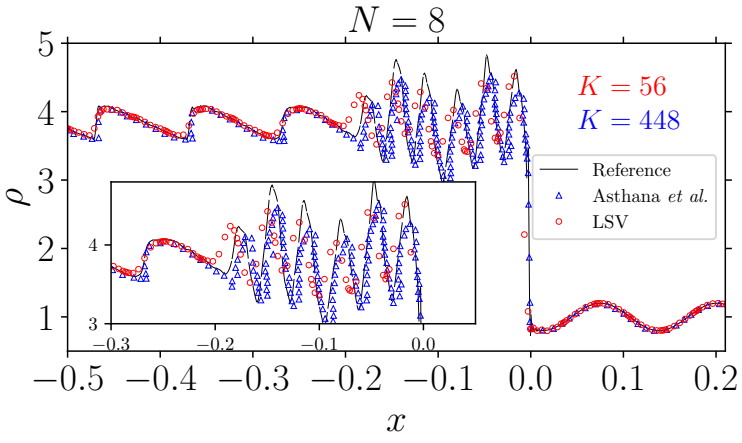
Asthana *et al.*, "Non-linear stabilization of high-order Flux Reconstruction schemes via Fourier-spectral filtering", J. Comp. Phys., 2015.

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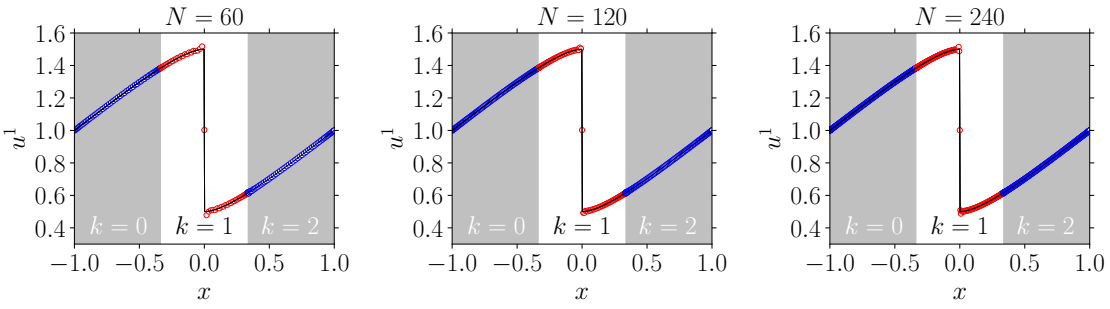
Shu-Osher interaction problem



Asthana *et al.*, "Non-linear stabilization of high-order Flux Reconstruction schemes via Fourier-spectral filtering", J. Comp. Phys., 2015.

Shock capturing closure of the fNS equations

Very high-order Burgers' problem



Shock/vortex interaction



Double Mach reflection



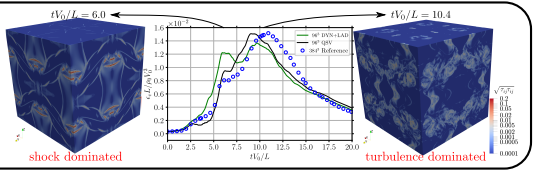
Double Mach reflection



Outline

Unified shock capturing and LES approach for finite-differences schemes

"*A unified Quasi-Spectral Viscosity (QSV) approach to shock capturing and large-eddy simulation*", J. Comp. Phys. (2022)



Filtered Navier-Stokes as a shock capturing method for flux reconstruction schemes

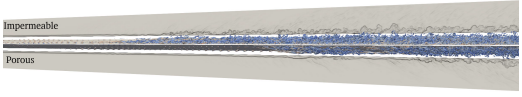
"*A Legendre spectral viscosity (LSV) method applied to shock capturing for high-order flux-reconstruction schemes*", J. Comp. Phys. (2022)



LES of hypersonic BL transition delay

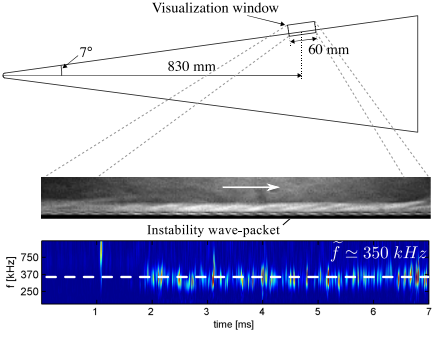
"*Numerical investigation of second-mode attenuation over carbon/carbon porous surfaces*", J. Spacecraft and Rockets (2019)

"*Delay of hypersonic transition to turbulence over broadband wall impedance*", Submitted to J. Fluid Mech.



Second Mode

- Principal transition mechanism for hypersonic flows ($M > 5$) in canonical geometries

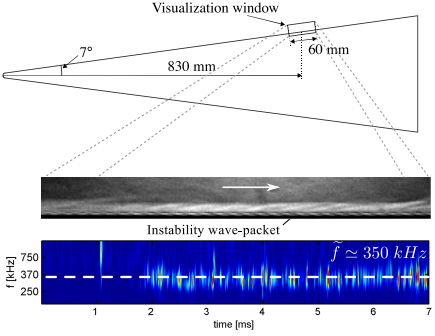


Wagner, PhD Thesis, 2014; Fedorov et al.: AIAA 2001-605.

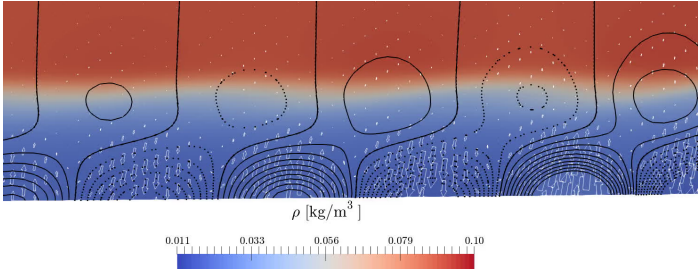
HBLs Transition Background

Second Mode

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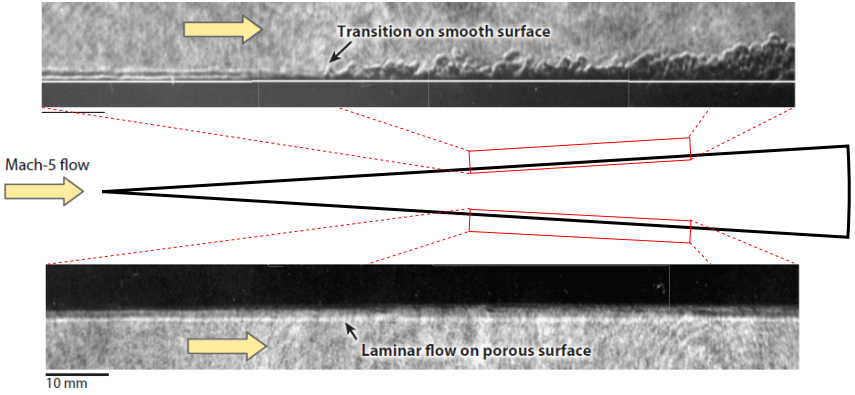
- Ultrasonic acoustic waves "trapped" below the sonic line



Wagner, PhD Thesis, 2014; Fedorov et al.: AIAA 2001-605.

Acoustic control of second-mode waves

- Can acoustic energy absorption be used for transition delay?

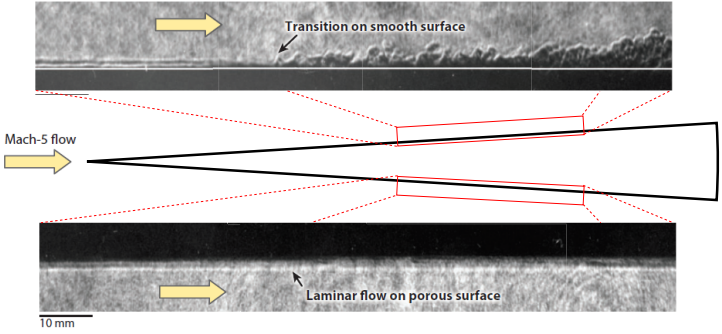


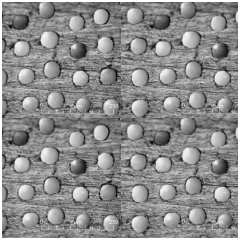
Rasheed et al.: AIAA 2002-481. Fedorov et al.: JFM 2003-99.

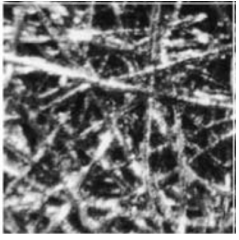
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Acoustic control of second-mode waves

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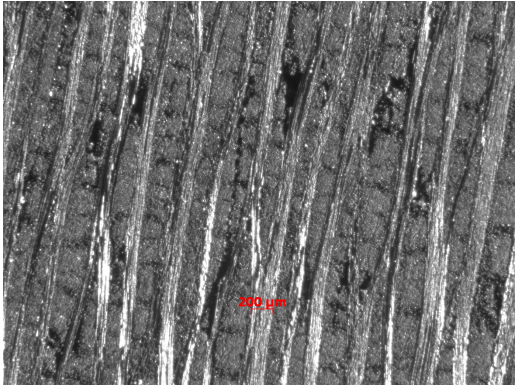


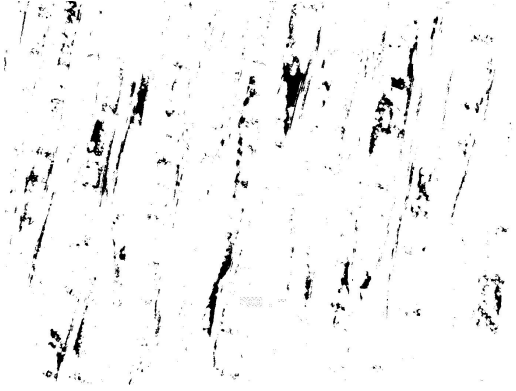


Rasheed et al.: AIAA 2002-481. Fedorov et al.: JFM 2003-99.

Carbon-fiber-reinforced carbon ceramics (C/C)

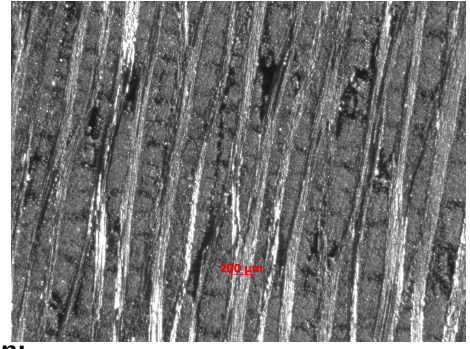
- Excellent thermal properties; can be made resistant to oxidation
- Inherent random porosity as a by-product of fabrication.

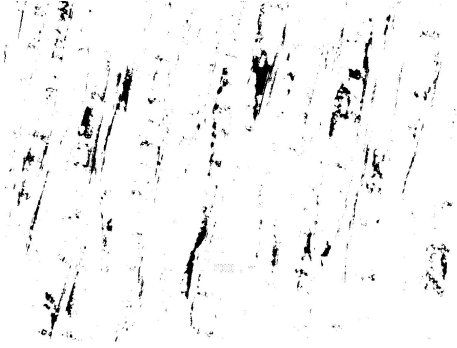




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Question:

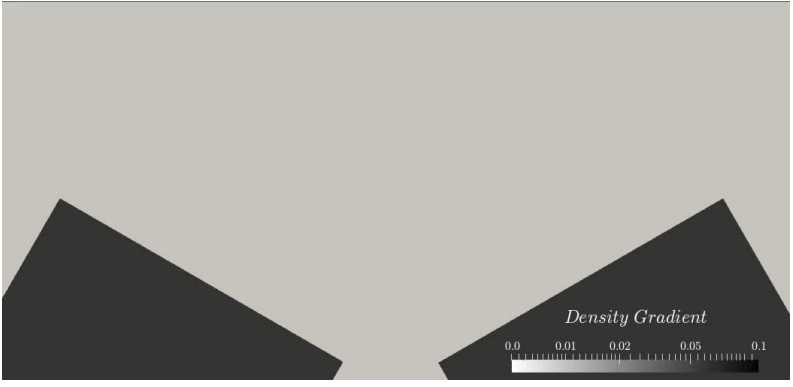
- Can the acoustic response of realistic C/C be accurately characterized and implemented in a high-fidelity simulation?

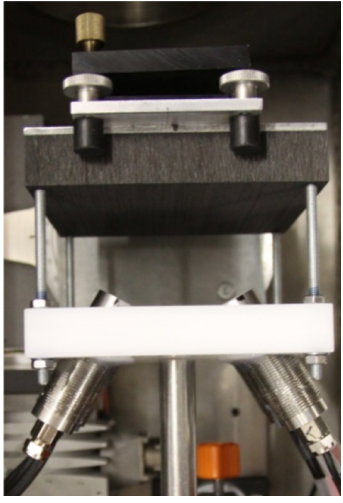
Impedance Characterization

PURDUE UNIVERSITY

Dr. Alex Wagner's benchtest experiments

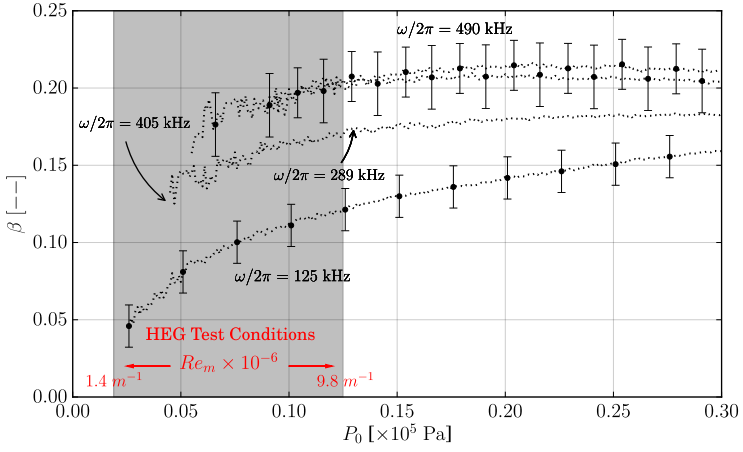
- Numerical Schlieren (left); benchtest setup (right)
- Max frequency is 0.5 MHz; Tranducers operate a discrete frequencies;
- Vacuum chamber allows pressure variation





Impedance Characterization

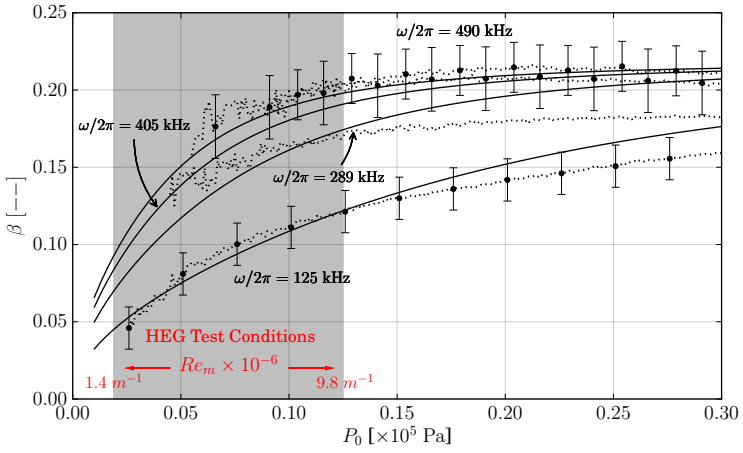
Pressure Domain Data



Impedance Characterization

PURDUE UNIVERSITY

Pressure Domain Data



Homogeneous Absorber Theory (HAT)

$$Z_{\mathrm{HAT}} = Z_\infty \frac{1 + e^{-j2k_a H}}{1 - e^{-j2k_a H}}$$

$$k_a = \frac{\omega}{a_0}\sqrt{\kappa}\sqrt{1 - j\frac{\omega_k}{\omega}}$$

$$Z_\infty = Z_0 \frac{\sqrt{\kappa}}{\phi}\sqrt{1 - j\frac{\omega_k}{\omega}}$$

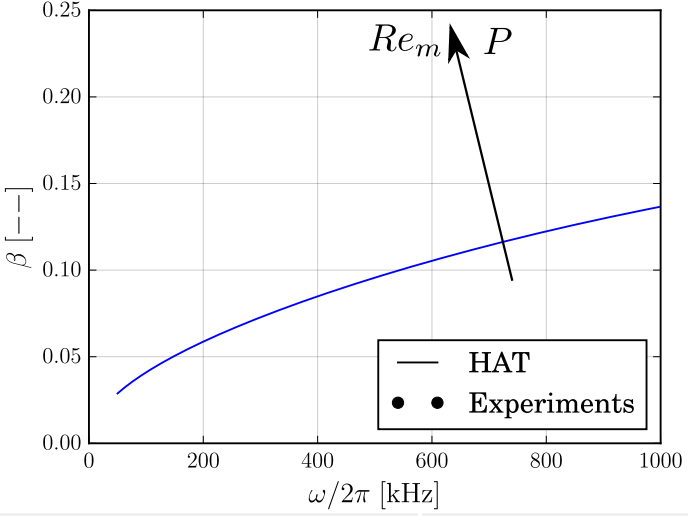
$$\omega_k = \frac{\Xi\phi}{\rho_0 \kappa}$$

- $H \rightarrow$ Depth
- $\phi \rightarrow$ Porosity
- $\Xi \rightarrow$ Flow Resistivity
- $\kappa \rightarrow$ Tortuosity

Impedance Characterization

PURDUE UNIVERSITY

Frequency Domain Data



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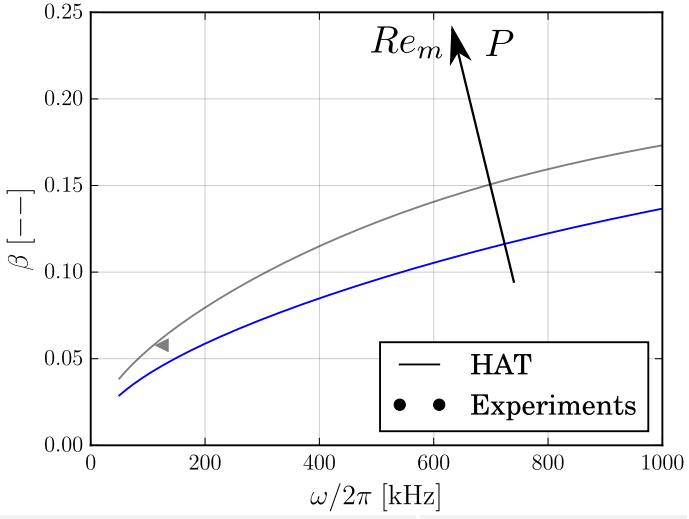
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Impedance Characterization

PURDUE UNIVERSITY

Frequency Domain Data



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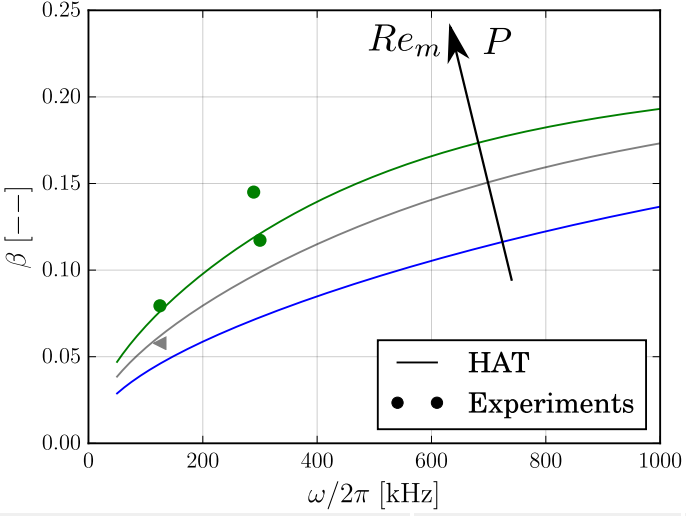
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Impedance Characterization

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Frequency Domain Data



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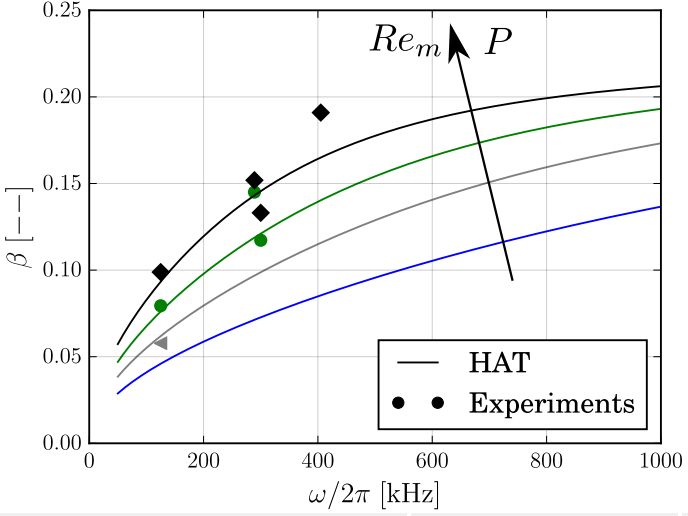
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Impedance Characterization

PURDUE UNIVERSITY

Frequency Domain Data



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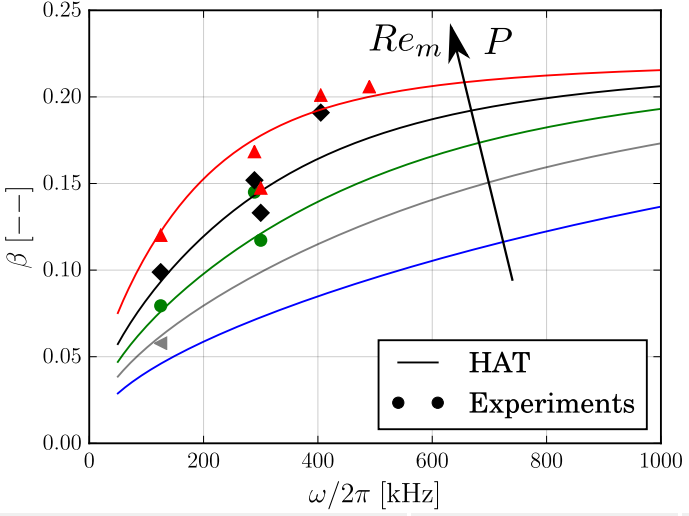
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Impedance Characterization

Frequency Domain Data



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Axisymmetric Stability Simulations

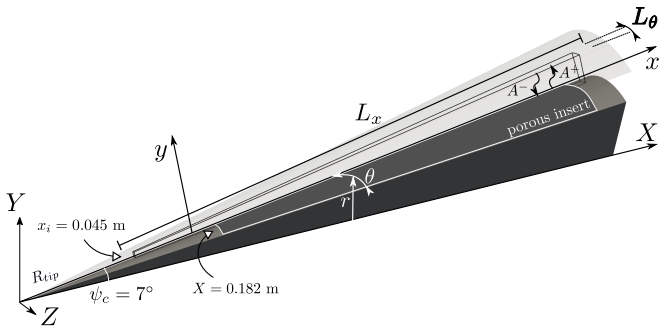
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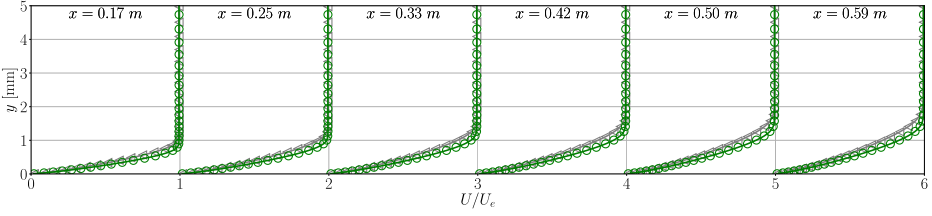
Sharp Cone

- Flow Parameters (Wagner *et al.* 2013):

Re_m [1/m]	$2.40 \cdot 10^6$	$4.06 \cdot 10^6$
M_∞ [−]	7.4	7.4
p_∞ [Pa]	1453	2129
T_∞ [K]	285	268
ρ_∞ [g/m^3]	17.7	27.6
L_x [m]	0.8	0.6
L_y [m]	0.07	0.05
Color/Symbol	◀	●

- Base Flow: Conical Blasius





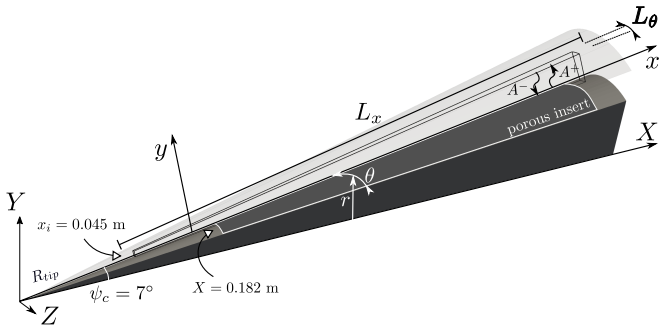
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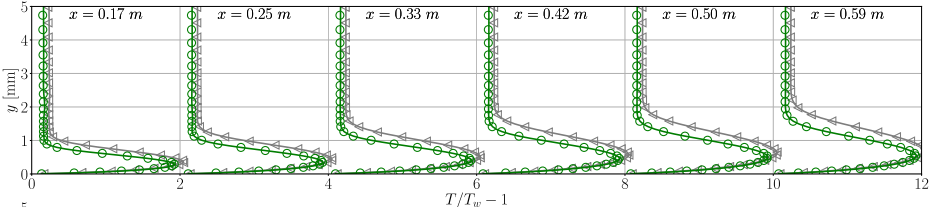
Sharp Cone

- Flow Parameters (Wagner *et al.* 2013):

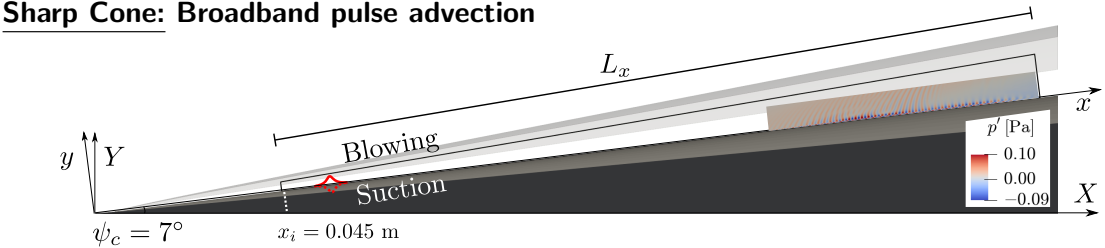
Re_m [1/m]	$2.40 \cdot 10^6$	$4.06 \cdot 10^6$
M_∞ [−]	7.4	7.4
p_∞ [Pa]	1453	2129
T_∞ [K]	285	268
ρ_∞ [g/m^3]	17.7	27.6
L_x [m]	0.8	0.6
L_y [m]	0.07	0.05
Color/Symbol	◀	●

- Base Flow: Conical Blasius





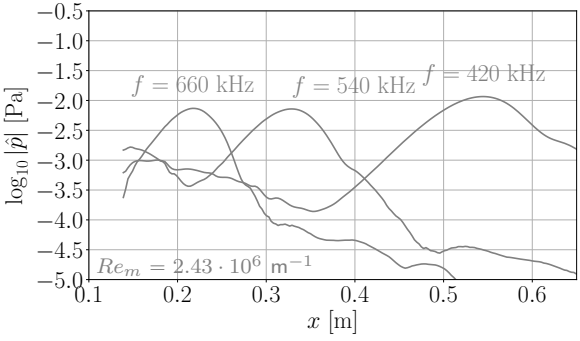
Sharp Cone: Broadband pulse advection

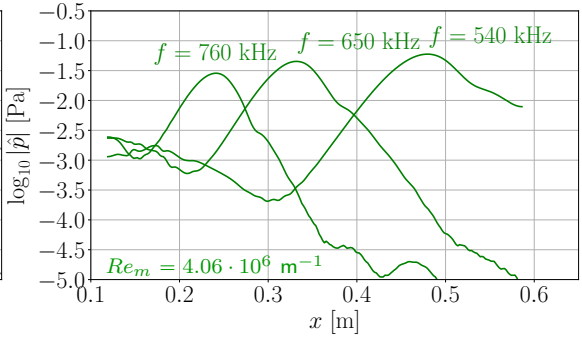




Sharp Cone: Broadband pulse advection

- QSV-LES: Impermeable wall (−)



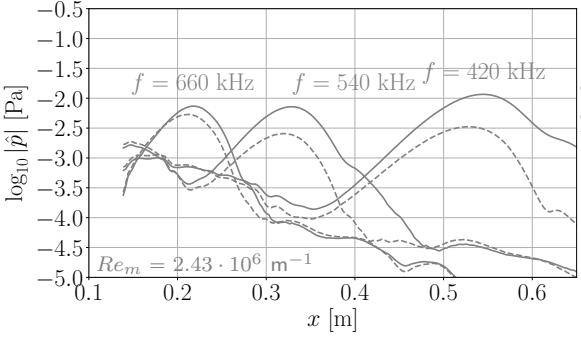


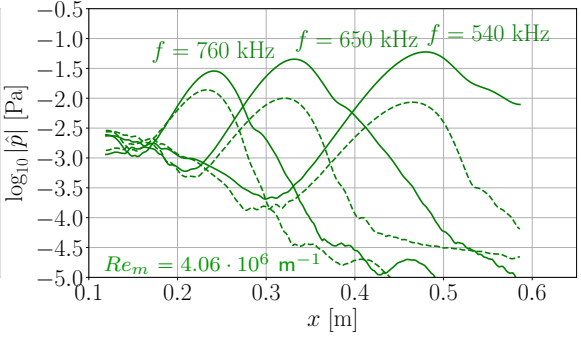
Scalo et al. PoF, 2015; Lin, Scalo, Hesselink JFM, 2016; Douasbin, Scalo, Selle, and Poinsot JCP, 2018

Axisymmetric Stability Simulations

Sharp Cone: Broadband pulse advection

- QSV-LES: Impermeable wall (−), Porous walls via TDIBC (−−)



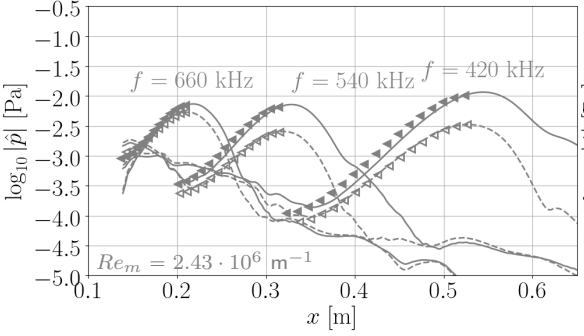


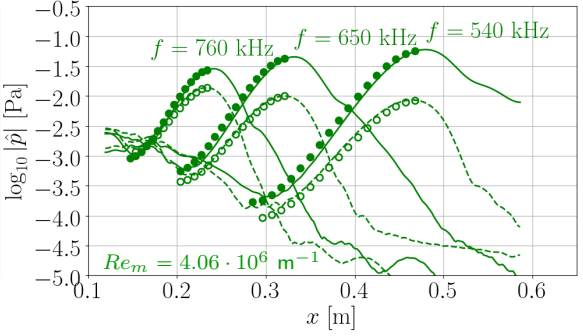
Scalo et al. PoF, 2015; Lin, Scalo, Hesselink JFM, 2016; Douasbin, Scalo, Selle, and Poinsot JCP, 2018

Axisymmetric Stability Simulations

Sharp Cone: Broadband pulse advection

- QSV-LES: Impermeable wall (−), Porous walls via TDIBC (−−)
- Laguerre-Galerkin LST: Impermeable wall (◀, ●), Impedance B.C. (◁, ○)





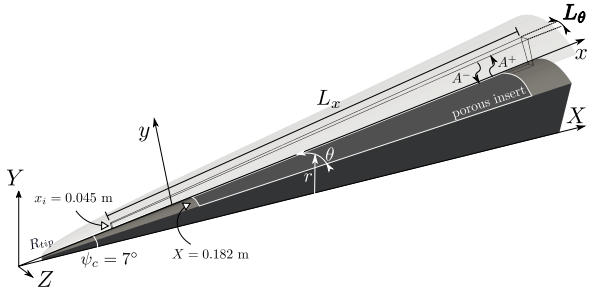
Scalo et al. PoF, 2015; Lin, Scalo, Hesselink JFM, 2016; Douasbin, Scalo, Selle, and Poinsot JCP, 2018

Axisymmetric Stability Simulations

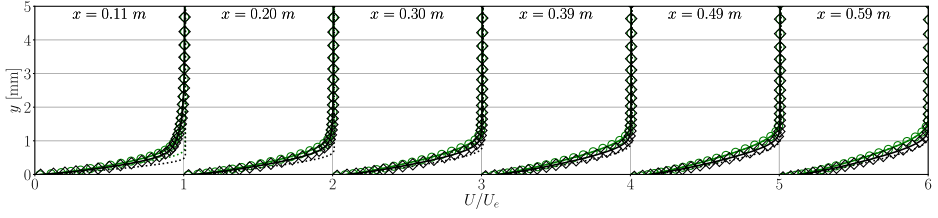
Blunt Cone

- Flow Parameters (Wagner *et al.* 2013):

Re_m [1/m]	$4.06 \cdot 10^6$	$6.43 \cdot 10^6$
M_∞ [−]	7.4	7.4
p_∞ [Pa]	2129	2129
T_∞ [K]	268	268
ρ_∞ [g/m^3]	27.6	27.6
L_x [m]	0.8	0.6
L_y [m]	0.07	0.05
Color/Symbol	●	◆



- Base Flow: DLR Flower precursor

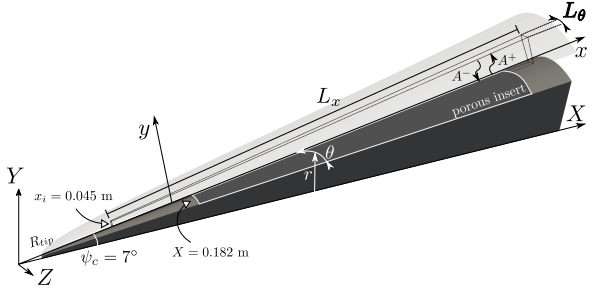


Axisymmetric Stability Simulations

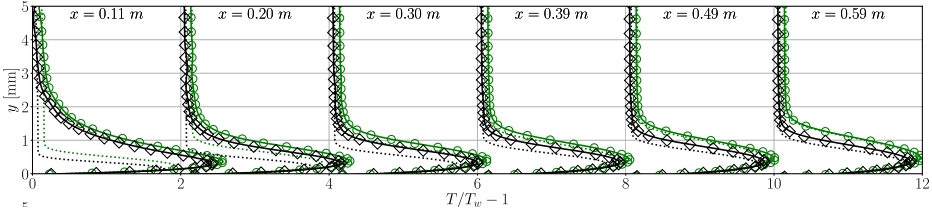
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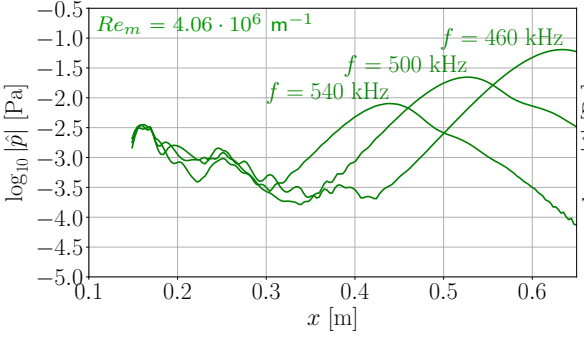


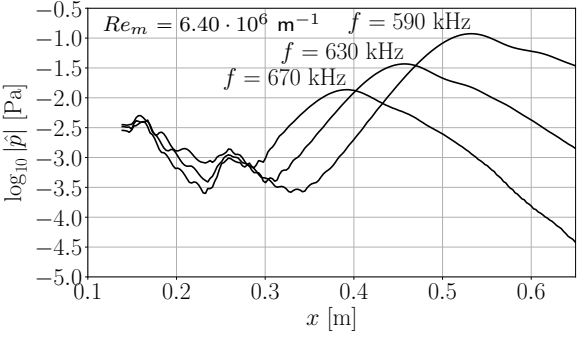
- Base Flow: DLR Flower precursor



Blunt Cone: Broadband pulse advection

- QSV-LES: Impermeable wall ($-$)



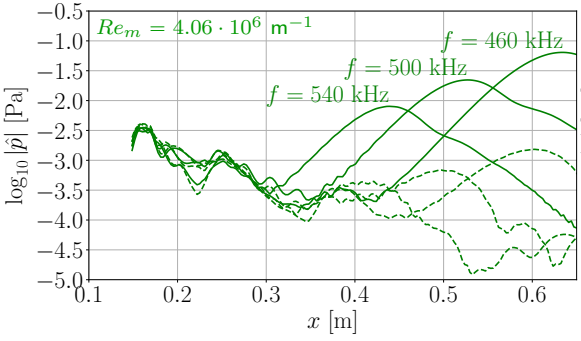


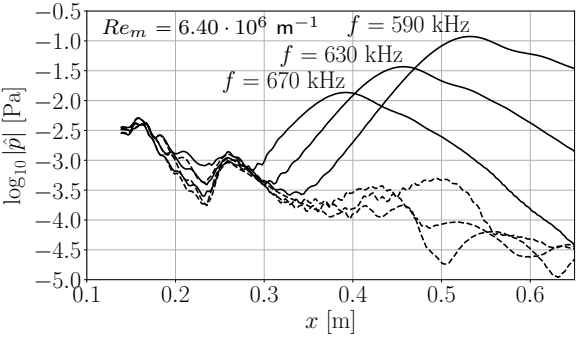
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Axisymmetric Stability Simulations

Blunt Cone: Broadband pulse advection

- QSV-LES: Impermeable wall (−), Porous walls via TDIBC (−−)





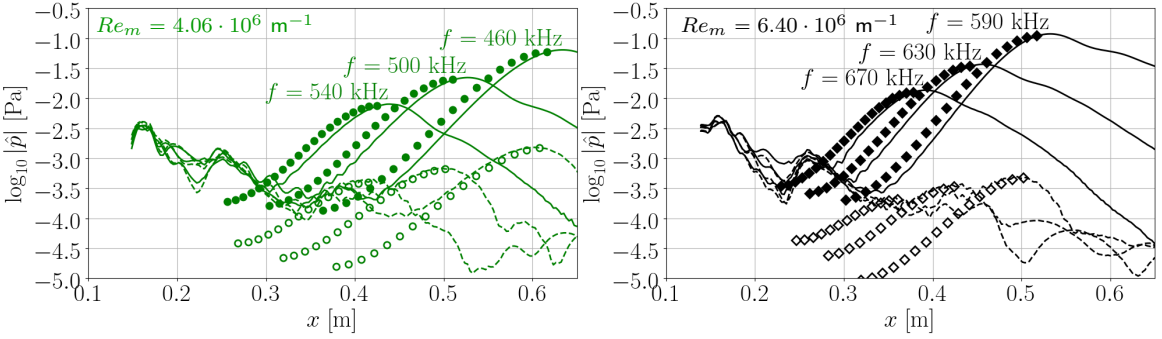
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Axisymmetric Stability Simulations

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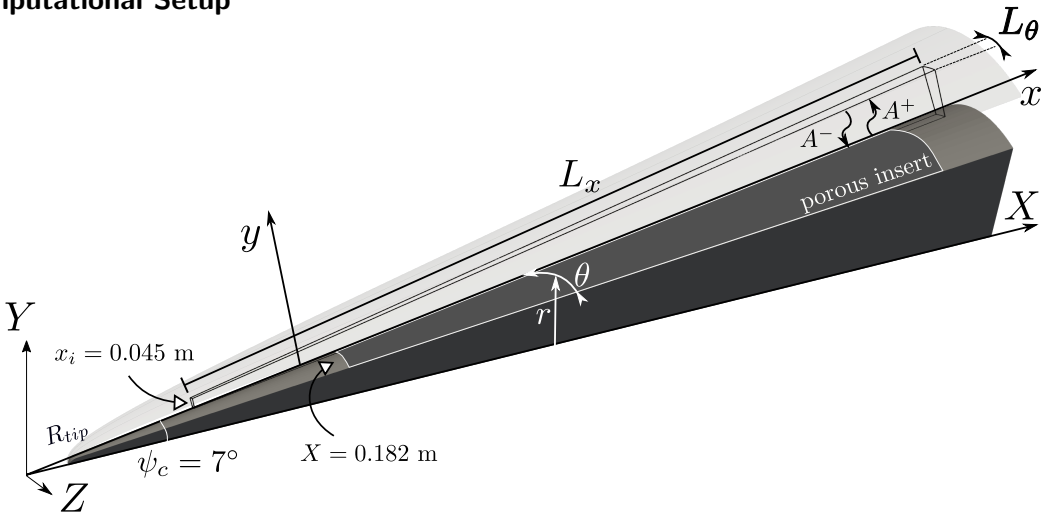
Blunt Cone: Broadband pulse advection

- QSV-LES: Impermeable wall (−), Porous walls via TDIBC (−−)
- Laguerre-Galerkin LST: Impermeable wall (●, ◆), Impedance BC (○, ◇)

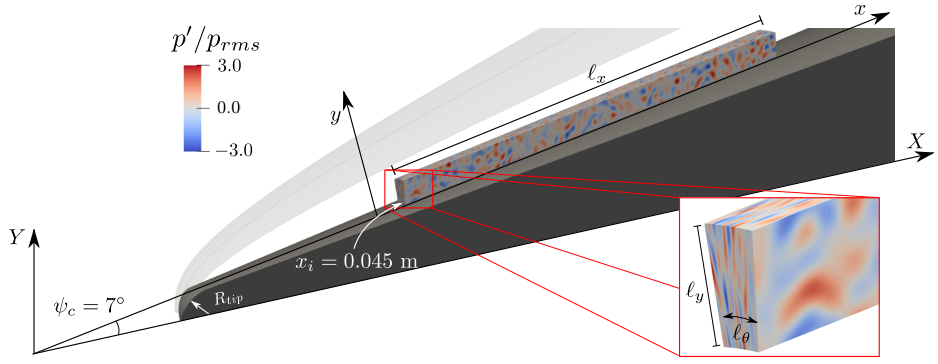


Scalo et al. PoF, 2015; Lin, Scalo, Hesselink JFM, 2016; Douasbin, Scalo, Selle, and Poinsot JCP, 2018

Computational Setup



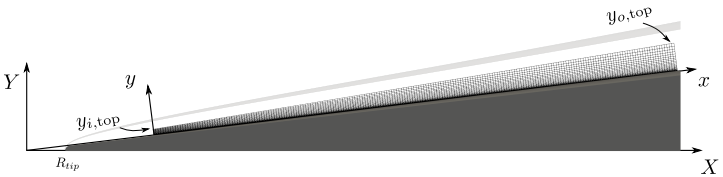
Volumetric Forcing

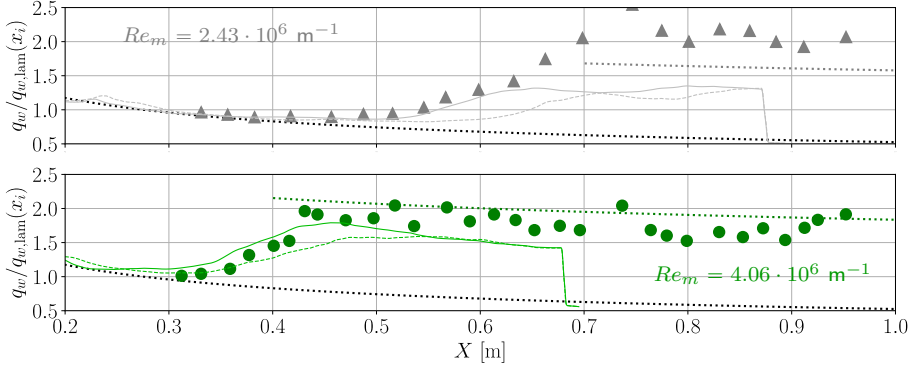


- Imposed noise in the transitional simulations follows the frequency distribution of freestream noise measured in the HEG wind Tunnel
- Forcing amplitude → $p_{rms}/p_\infty = 3\%$

Sharp: Grid Sensitivity

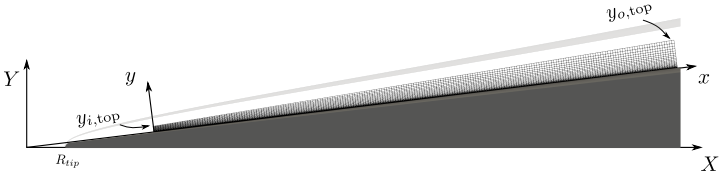
- Thin – $n_x = 2048; n_y = 96; n_z = 48$

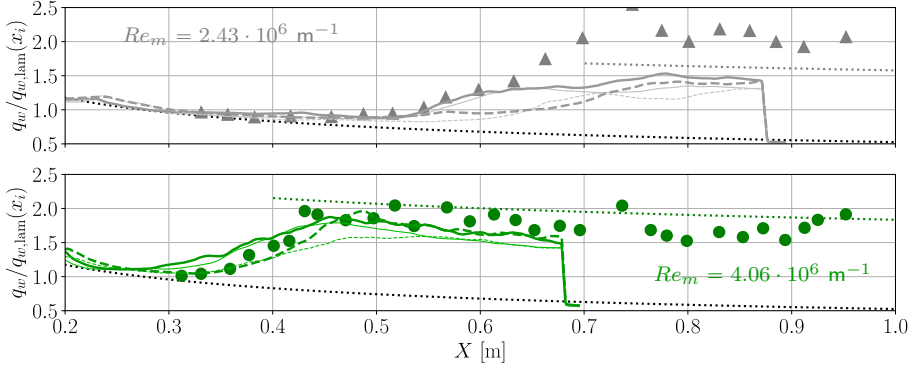




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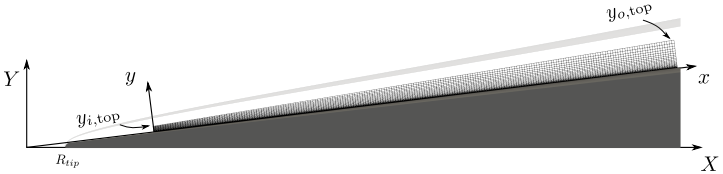
- Thin – $n_x = 2048; n_y = 96; n_z = 48$
- Inter. – $n_x = 2560; n_y = 120; n_z = 64$

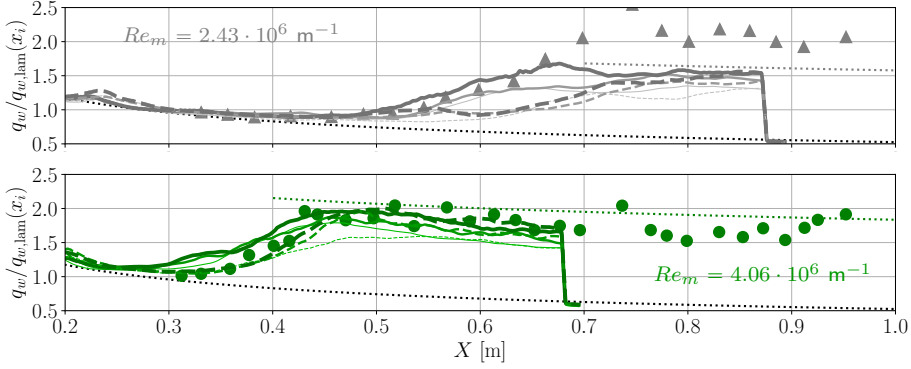




Sharp: Grid Sensitivity

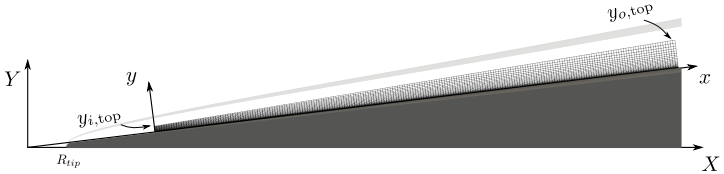
- Thin – $n_x = 2048; n_y = 96; n_z = 48$
- Inter. – $n_x = 2560; n_y = 120; n_z = 64$
- Thick – $n_x = 3072; n_y = 144; n_z = 80$

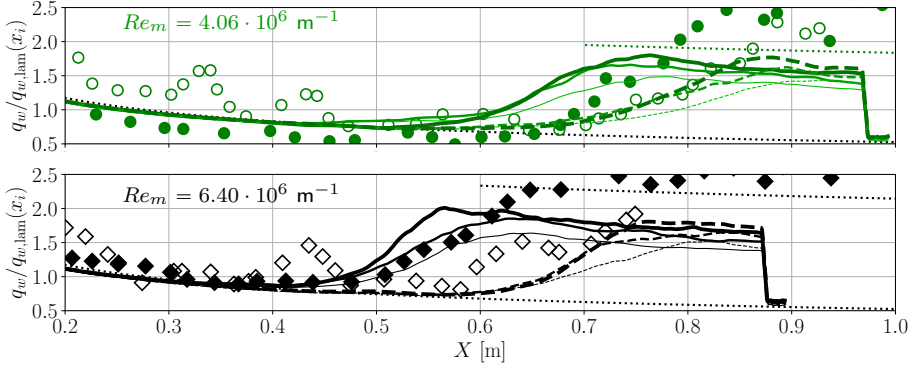




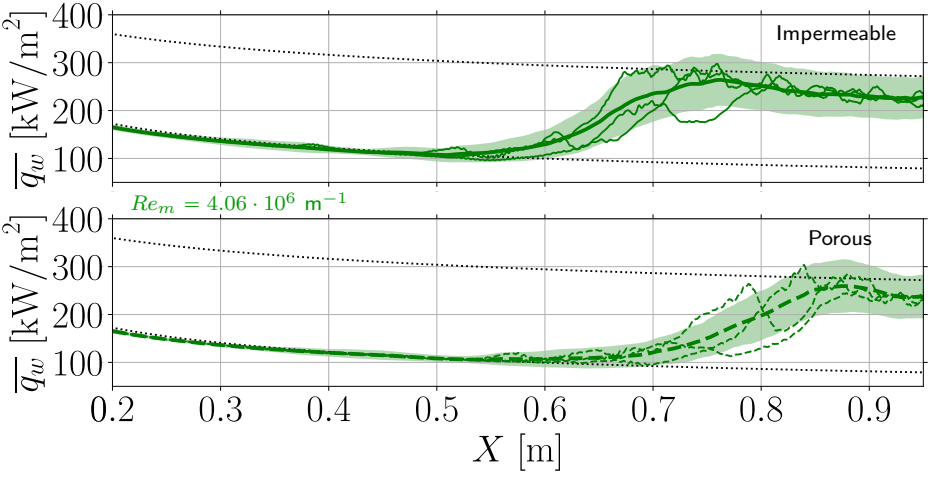
Blunt: Grid Sensitivity

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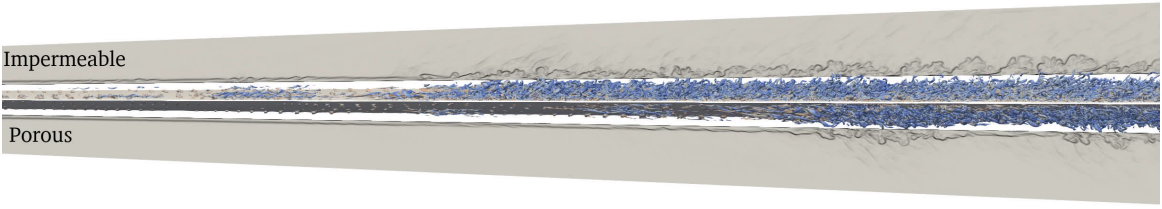




Blunt: Turbulent spots

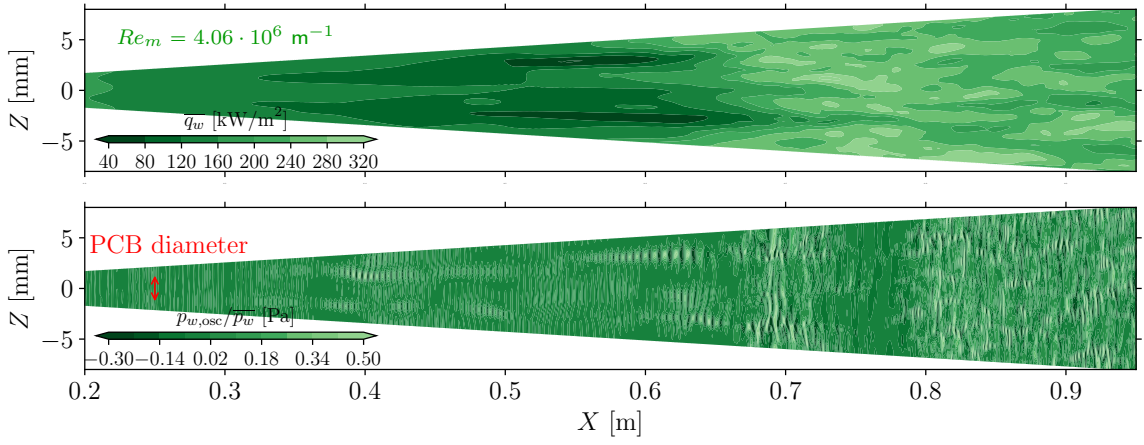


Blunt: Turbulent spots

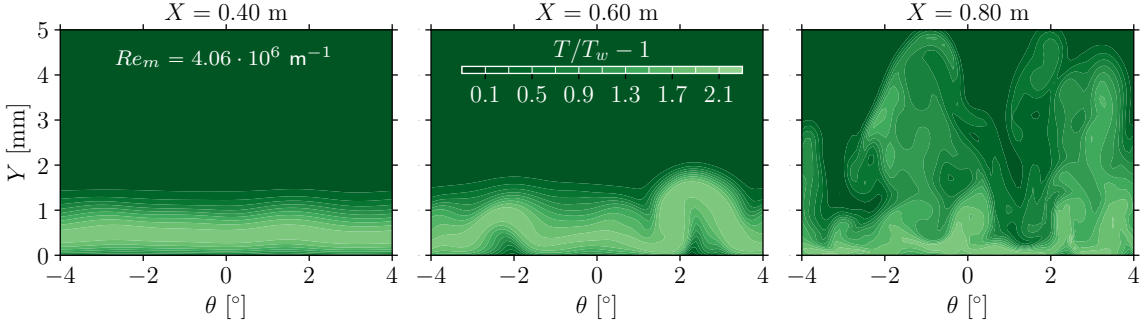


$Re_m = 4.06 \cdot 10^6 \ \mathrm{m}^{-1}$

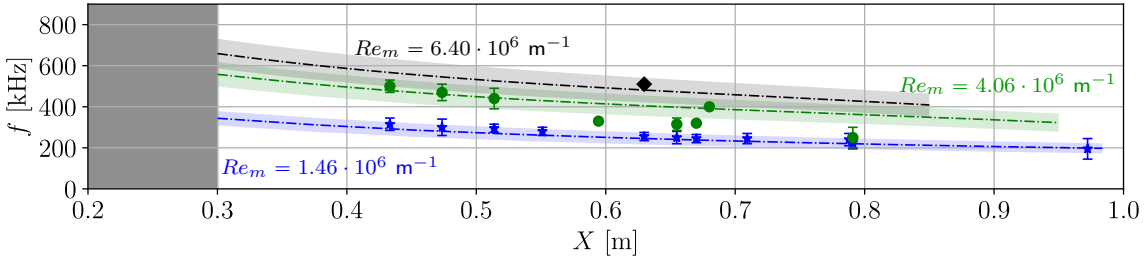
Blunt: Frequency content



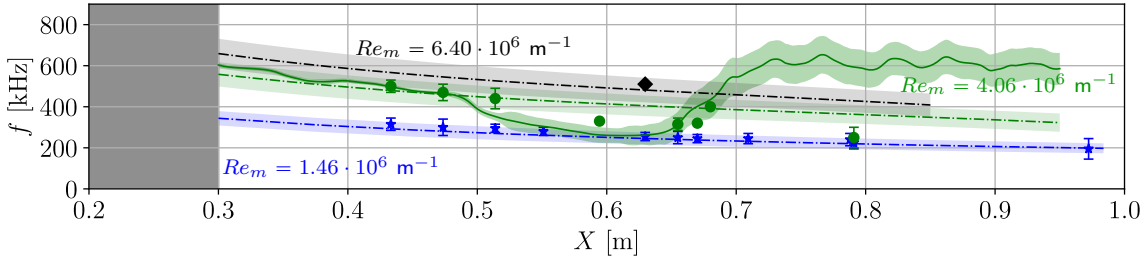
Blunt: Frequency content



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Blunt: Frequency content

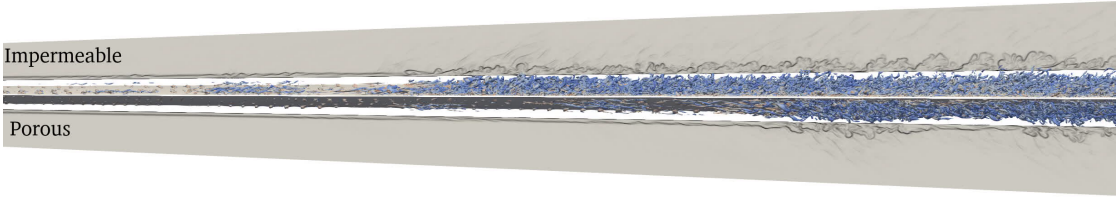


Conclusion

- QSV algorithm successfully extends LES framework from turbulent flows only to shock dominated flows and their interaction for finite-differences schemes

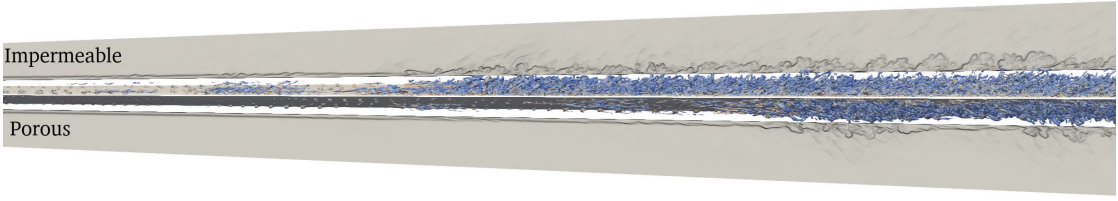
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- QSV was works well for hypersonic transitional boundary layer simulations, showcasing its dynamic property



Conclusion

- QSV algorithm successfully extends LES framework from turbulent flows only to shock dominated flows and their interaction for finite-differences schemes
- QSV was works well for hypersonic transitional boundary layer simulations, showcasing its dynamic property



- LSV is the first step of extending the the unified shock capturing/subfilter turbulence modeling framework to unstructured solvers

Thank you

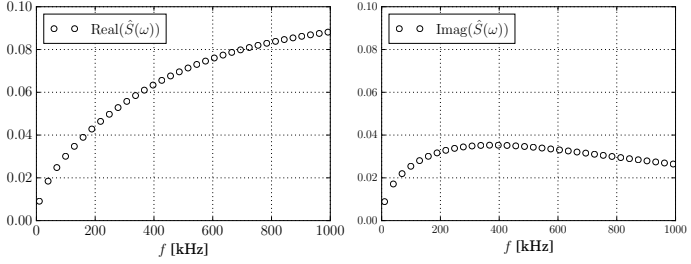
Extra Slides

Time-Domain Impedance Boundary Conditions (TDIBC)

- Acoustic impedance ($\hat{p} = Z(\omega)\hat{v}$) imposed via multi-pole (or multi-oscillator) expansion, via complex poles (p_k) and residues (μ_k), of the softness coefficient:

$$\widehat{S}(\omega) = \frac{2}{1 + Z_*(\omega)} \approx \sum_{k=1}^{n_o} \left[\frac{\mu_k}{s - p_k} + \frac{\mu_k^*}{s - p_k^*} \right],$$

- Poles and residues are obtained by Vector fitting technique described by Gustavsen (1999).



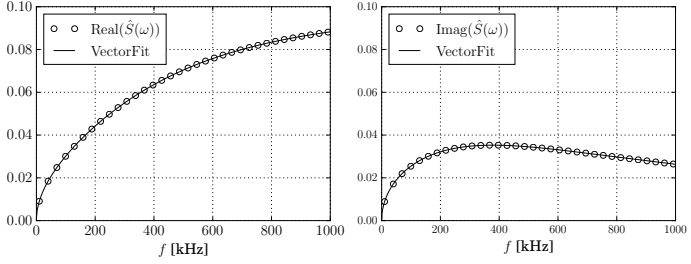
Fung and Ju, 2001,2004; Scalo et al. PoF, 2015; Lin, Scalo, Hesselink JFM, 2016; Douasbin, Scalo, Selle, and Poinsot JCP, 2018

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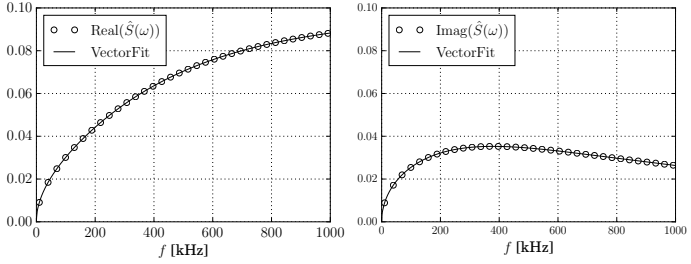
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Time-Domain Impedance Boundary Conditions (TDIBC)

- The reflected wave $A^+(t)$ is calculated as a causal convolution integral of the incident wave A^-

$$A^+(t) = -A^-(t) + \int_0^\infty S(\tau) A^-(t-\tau) d\tau$$

$$v'(t) = \frac{1}{2}\left[A^-(t) + A^+(t)\right]$$



Fung and Ju, 2001,2004; Scalo et al. PoF, 2015; Lin, Scalo, Hesselink JFM, 2016; Douasbin, Scalo, Selle, and Poinsot JCP, 2018

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If you rerun the document (without altering it) this surplus page will go away, because LaTeX now knows ho
to expect for this document.