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Synthesizing Turbulence?

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Synthesizing Turbulence?

From Reynolds to Wiener We Ask:

Is there a closed-form Solution
to
Steady PPF Turbulence?

Maybe!

Yaglom and Lumley Say:

"We believe that, even after 100 years, turbulence studies are still in their infancy. We are naturalists, observing butterflies in the wild. We are still discovering how turbulence behaves, in many respects. We do have a crude, practical, working understanding of many turbulence phenomena, but certainly nothing approaching a comprehensive theory, and nothing that will provide predictions of an accuracy demanded by engineers."

Yaglom and Lumley, "A Century of Turbulence, 2001"

A Somber Reflection

What Do We Have?

Current ways to deal with turbulence:

- Log/Wall/Colebrook, dimensional analysis
- Closure Formulas: 1-point, $\kappa-\varepsilon$, etc.
- DNS with periodic boundaries
- Lattice-Boltzmann
- Wiener Functionals

Today cannot Synthesize vs. simulate a DNS run

What Do We Want?

A **Closed-Form Equation** to **Synthesize** PPF flow:

$u^i(x,y,z,t)=\{\text{closed form formula}\}$

Using a few nice functions
AND Wiener Convolution Integrals

Generate the Complete Velocity Field

Today I want to demonstrate how this can be done.

Basis: *Plane Poiseuille Flow*

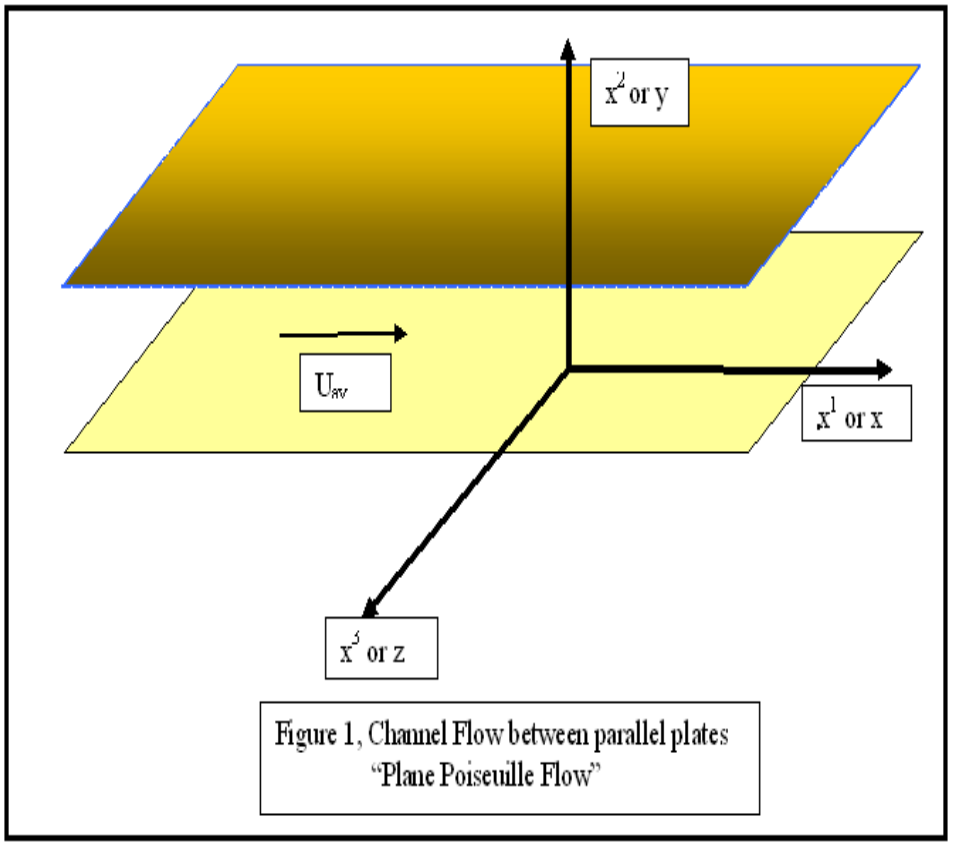


Figure 1, Channel Flow between parallel plates
"Plane Poiseuille Flow"

Usual Assumptions, Same as DNS

- Newtonian Fluid - Constant Viscosity
- Continuum Equations
  - Conservation of Mass
  - Navier Stokes
- Incompressible Fluid – Constant Density
- No-Slip conditions at the Walls
- Steady Flow
  - Fully Developed
  - No intermittency
  - No Cavitation
- Reynolds Averaging Applies
  - Stationary on x, t, and z
  - Ergodic on x, t, and z

Continuum Foundation

- Navier-Stokes Equations with Incompressible Newtonian Fluid

$$\frac{\partial}{\partial t}u^i\left(x,y,z,t\right)+u^k\nabla^k u^i-\frac{1}{R_e}\nabla^2 u^i+\nabla^i p=F^i$$

$$\nabla^k u^k\left(x,y,z,t\right)=0$$

- Smooth Walls, No External "Randomness"
- MFD Equations include Bullard's Equation

$$\frac{\partial}{\partial t}b^i\left(x,y,z,t\right)+u^k\nabla^k b^i-\frac{1}{R_m}\nabla^2 b^i=b^k\nabla^k u^i$$

$$\nabla^k b^k\left(x,y,z,t\right)=0$$

$$F^i=\frac{M^2}{R_e}b^k\left(u^k b^i-b^k u^i\right)$$

Start With Velocities

The Solution We Seek is:

$u^i(x,y,z,t)$

a DNS run is an example of the collection of Solutions

We choose to represent the ***Ensemble*** of solutions as:

$u^i(x,y,z,t,\alpha)$ including the α parameter

And all averages are of the form:

$$\overline{foo(y)} = \int_0^1 foo(x,y,z,t,\alpha)\,d\alpha$$

So $u^i(x,y,z,t,\alpha)$ is just a real function of 5 variables!

What Now?

- Standard Basis so far – No Surprises

- Develop The Process in Three Steps:
  - $U(t,\alpha)$ – A simple stationary RV in 1D
    A single velocity at a fixed point in the Channel
  - $U^x(x,y,t,\alpha)$ – A Solution for 2D PPF
    The streamwise velocity in 2D PPF
  - $U^{x,z}(x,y,z,t,\alpha)$ – A Solution for 3D PPF
    Streamwise and spanwise velocities in 3D PPF

1D: U(t,α) A Stationary RV

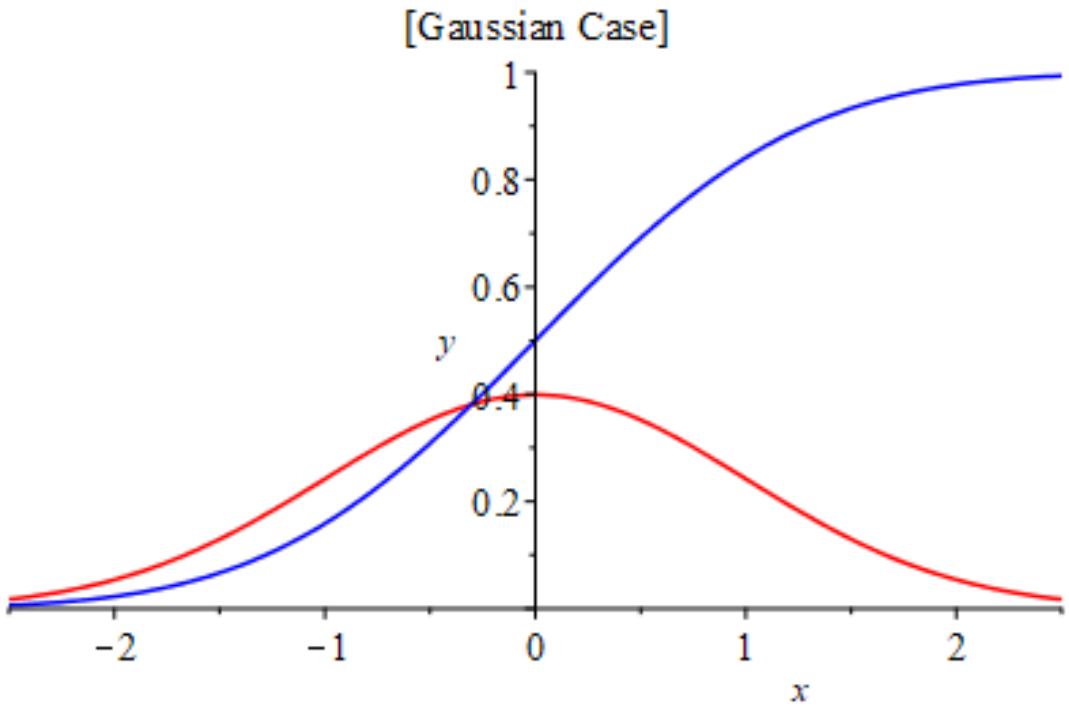
- A single velocity at a fixed point in the Channel
  - Stationary and Ergodic on "t"
  - Bounded Mean and Variance
  - Has as many derivatives as we want

- Has a cdf – cumulative distribution function

$$\mathcal{Pr}\left\{u(t,\alpha) \le \lambda\right\} = \mathrm{cdf}^{u}(\lambda) \quad \text{stationary, independent of "t"}$$

  - Smooth, monotone strictly increasing
  - Derivative is pdf – probability density function

The Normal cdf and pdf



The Associated Quantile RV

- The associated quantile function for U(t,α) is:

$$q(t,\alpha) = \mathrm{cdf}^{u}\left(u(t,\alpha)\right)$$

And its Inverse

$$u(t,\alpha) = \mathrm{icdf}^{u}\left(q(t,\alpha)\right)$$

The Associated Quantile RV

- The associated quantile function for U(t,α) is:

$$q(t,\alpha) = \mathrm{cdf}^{u}\left(u(t,\alpha)\right)$$

$$u(t,\alpha) = \mathrm{icdf}^{u}\left(q(t,\alpha)\right)$$

- What is the $\mathrm{cdf}^{q}(..)$ of q?

$$\mathcal{Pr}\{u(t,\alpha) \le \lambda\} = \mathrm{cdf}^{u}(\lambda)$$

$$\mathcal{Pr}\{\underbrace{\mathrm{cdf}^{u}(u(t,\alpha))}_{=q(t,\alpha)} \le \underbrace{\mathrm{cdf}^{u}(\lambda)}_{\triangleq\mu}\} = \underbrace{\mathrm{cdf}^{u}(\lambda)}_{\triangleq\mu}$$

The Associated Quantile RV

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- The Uniform Distribution, Mirabile Dictu!

$$\mathcal{Pr}\{q(t,\alpha) \le \mu\} = \mu$$

The Associated Normal RV

- The Normal cdf and its inverse are:

$$\mathrm{cdf}^{N}(\lambda) = \tfrac{1}{2}\left[1 + \mathrm{erf}\left(\tfrac{1}{\sqrt{2}}\lambda\right)\right] \quad \text{The "Normal" Distribution}$$

$$\mathrm{icdf}^{N}(\mu) = \sqrt{2}\cdot \mathrm{ierf}\left(2\mu - 1\right) \quad \text{The "Normal" Quantile Function}$$

- Then the Associated Normal RV is:

$$\begin{aligned}\phi(t,\alpha) &= \mathrm{icdf}^{N}\left(\ q(t,\alpha)\ \right)\\ &= \mathrm{icdf}^{N}\left(\ \mathrm{cdf}^{u}\left(\ u(t,\alpha)\ \right)\ \right)\\ &= M_{u2\phi}\left(u(t,\alpha)\right)\end{aligned}$$

$$M_{u2\phi}(\lambda) \triangleq \mathrm{icdf}^{N}\left(\ \mathrm{cdf}^{u}(\lambda)\ \right)$$

The Associated Normal RV

- The original $u(t,\alpha)$ and $\phi(t,\alpha)$ map one-to-one by:

$$u(t,\alpha) = M_{\phi 2u}\left(\phi(t,\alpha)\right)$$

$$M_{\phi 2u}(\lambda) = \text{icdf}^{u}\left(\text{cdf}^{N}(\lambda)\right)$$

$$\phi(t,\alpha) = M_{u2\phi}\left(u(t,\alpha)\right)$$

$$M_{u2\phi}(\lambda) = \text{icdf}^{N}\left(\text{cdf}^{u}(\lambda)\right)$$

- $M_{u2\phi}(\lambda)$ and $M_{\phi 2u}(\lambda)$ are both continuous, strictly increasing, independent of "t", infinite range and domain

More About This Normal RV

- $\phi(t,\alpha)$ is Normal, Stationary, Ergodic, Mean=0, σ=1
- All statistics are in **cfd^{u}** and **$M_{u2\phi}/M_{\phi 2u}$**
- A Stationary Gaussian RV is Completely characterized by Mean and Autocorrelation (Yaglom, Wikipedia)
- So, how do we generate this $\phi(t,\alpha)$?
  - Use the Wiener Process
  - In a Convolution Integral

The Wiener Process

- "Brownian Motion" mapped to α yields $r(t,\alpha)$, Wiener's "Random Function of Time and Phase"
- Many Ways to create $r(t,\alpha)$
- "Wiener Measure" – his Construction is Simple and Instructive
- Historical Note – Wiener's Little Book, "Nonlinear Problems in Random Theory" is seminal.
- Wiener's 1938 Paper Suggested This Results

About the Wiener Process

- r(s,α) is continuous on s, not bounded in variation, fractal dimension of 3/2
- It has Gaussian distribution with independent increments and variance equal to "s"
- r(s,α) is integrable on α, and Expected Values are Integrals on α over [0,1]
- Again, it is continuous (not almost), but not bounded variation everywhere on s

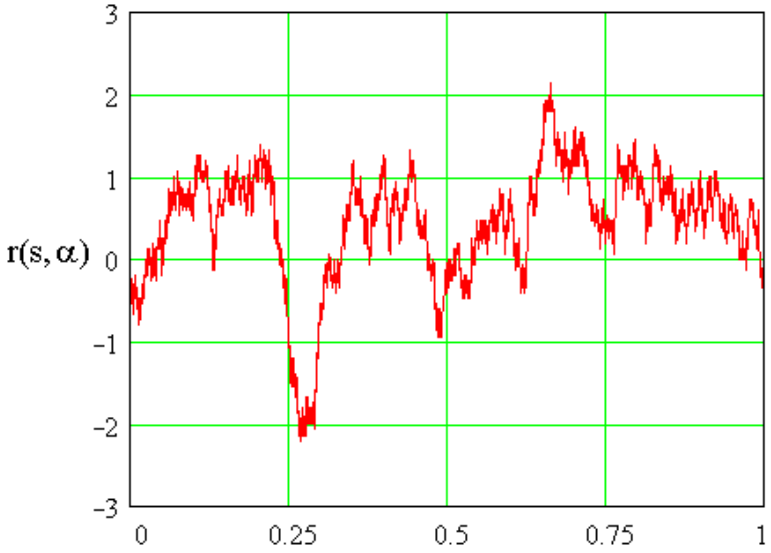


Figure 1, A Sample of the Wiener Process

The Wiener Integral

- The Wiener Stochastic Integral is

$$\phi(t,\alpha) = \int \Phi(t-s)\, dr(s,\alpha) \qquad \text{Usually over infinite limits}$$

- The Integral exists because $\Phi(t)$ has bounded variation and $r(s,\alpha)$ is continuous

- $\phi(t,\alpha)$ is an extraordinary Random Function
  - Stationary and Ergodic on "t"
  - "Isometric", i.e.

$$\int_0^1 f(t,\alpha)\cdot g(t+s,\alpha)\cdot d\alpha = \int_{-\infty}^{+\infty} F(t)\cdot G(t+s)\cdot dt$$

  - Gaussian, Differentiable, and Invertible

- Perfect for the Normal RV We need

Connecting The Two

- Now Construct u(t,a) by

$$u(t,\alpha) = M_{\phi 2u}\left(\phi(t,\alpha)\right)$$

$$M_{\phi 2u}(\lambda) = \text{icdf}^{u}\left(\text{cdf}^{N}(\lambda)\right)$$

$$\phi(t,\alpha) = \int \Phi(t-s)\, dr(s,\alpha)$$

- We have complete knowledge of u(t,a) with two functions:
  - $M_{\phi 2u}(\lambda)$ continuous, strictly increasing, infinite range and domain
  - $\Phi(s)$ a Normal autocorrelation function
- Both Functions Readily Measurable from Experiment
- All Statistics, Realizations, and Visualizations are readily constructed

Pause to Reflect on 1D RV

- Hard to Overstate the Significance of this Result
  - We can **Synthesize** (vs Simulate) Any Stationary RV
- Looking Ahead:
  - We can extend this 1D result to 2D and 3D PPF
    - More Complex in Detail, but no more complex in Principle
  - We can Suggest ways to Analytically Derive the Functions
    - Perhaps Ritz-Galerkin Methods
    - Perhaps Metalog Models for cdf(..)
- Extension to Other Flows
  - MFD, HIT, CPF – Any Stationary Flow

Incompressible 2D PPF

- One Independent Velocity $u^x(x,y,t,\alpha)$ because:

$$\nabla \cdot u = 0$$

$$u^y = -\int \partial_x u^x \cdot dy$$

- The $cdf^x(\lambda,y)$ depends on y, becoming sharper at the walls
- The Associated quantile and Normal RVs are then:

$$q^x(x,y,t,\alpha) = cdf^x\left(u^x(x,y,t,\alpha),\ y\right)$$

$$\phi^x(x,y,t,\alpha) = M^x_{u2\phi}\left(u^x(x,y,t,\alpha),\ y\right)$$

$$M^x_{u2\phi}(\lambda,\ y) = icdf^N\left(cdf^x(\lambda,\ y)\right)$$

- Recover $u^x(x,y,t,\alpha)$ with:

$$u^x(x,y,t,\alpha) = M^x_{\phi 2u}\left(\phi^x(..),\ y\right)$$

$$M^x_{\phi 2u}(\lambda,\ y) = icdf^x\left(cdf^N(\lambda,\ y)\right)$$

Constructing $\phi(x,y,t,\alpha)$

- The Appropriate Wiener Integral is:

$$\phi(x,y,t,\alpha) = \int \Phi(x-\varepsilon, y, t-\tau)\, dr(\varepsilon,\tau,\alpha)$$

  - Surface Integral over $\{\xi,\tau\}$ space
  - $dr(\xi,\tau,\alpha)$ is a surface element, variance is area of element
  - $r(\xi,\tau,\alpha)$ is continuous on $\{\xi,\tau\}$
  - $\Phi(x,y,t)$ is a continuous functions with bounded variation
- This can be Made Quite Rigorous
- Experimental Results readily determine $\Phi(x,y,t)$
- Both $cdf^x(\lambda,y)$ and $\Phi(x,y,t)$ are solutions of PDE's
  - Non-Random Solution to Equations of Motion
  - Once Known, they are known for all realizations

Pause to Reflect on 2D PPF

- With $cdf^x(\lambda,y)$ and $\Phi(x,y,t)$:
  - We can **Synthesize** (vs Simulate) Any 2D PPF Realization
  - Compute any moment, correlation, energy spectrum, …
  - Visualize any Velocity, Vorticity, …
- Looking Ahead:
  - We can extend this 2D result to 3D PPF
    - More Complex in Detail, but no more complex in Principle
  - We can Suggest ways to Analytically Derive the Functions
    - Perhaps Ritz-Galerkin Methods
    - Perhaps Metalog Models for cfd(..)
- Extension to Other Flows
  - MFD, HIT, CPF – Any Stationary Flow

Incompressible 3D PPF

- A VERY Quick Look with Little Explanation
- Two Independent Velocity $u^k(x,y,z,t,\alpha)$ (k=x or z) because:

$$\nabla \cdot u = 0$$

$$u^y = -\int \left(\partial_x u^x + \partial_z u^z \right) \cdot dy$$

- The $cdf^k(\lambda,y)$ depends on y, becoming sharper at the walls
- The Associated quantile and Normal RVs are then:

$$q^k(x,y,z,t,\alpha) = cdf^k\left(u^k(x,y,z,t,\alpha),\ y\right)$$

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- Recover $u^k(x,y,t,\alpha)$ with:

$$u^k(x,y,t,\alpha) = M^k_{\phi 2u}\left(\phi^k(..),\ y\right)$$

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Constructing $\phi^k(x,y,z,t,\alpha)$

- The Appropriate Wiener Integral is:

$$\phi^k(x,y,z,t,\alpha) = \int \Phi^{k\beta}(x-\varepsilon, y, z-\zeta, t-\tau)\, dr^\beta(\varepsilon,\zeta,\tau,\alpha)$$

  - Volume Integral over $\{\xi,\zeta,\tau\}$ space
  - $dr^\beta(\xi,\zeta,\tau,\alpha)$ is a volume element, variance is volume of element
  - $r^\beta(\xi,\zeta,\tau,\alpha)$ is continuous on $\{\xi,\zeta,\tau\}$ independent for each "β"
  - $\Phi^{k\beta}(x,y,z,t)$ are 4 continuous functions with bounded variation
- This can be Made Quite Rigorous
- Experimental Results readily determine $\Phi^{k\beta}(x,y,z,t)$
- Both $cdf^k(\lambda,y)$ and $\Phi^{k\beta}(x,y,z,t)$ are solutions of PDE's
  - Non-Random Solution to Equations of Motion
  - Once Known, they are known for all realizations
- Much more to be said!

Conclusions

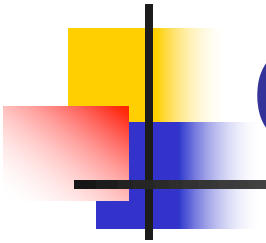
- The Question is this:

  Is There A Solution to PPF?

- The Answer we Proffer is this:

  Yes! With Ordinary Functions and the Wiener Integral

- The Details:
  - Cumulative Distribution Function is Central
  - Riemann-Stieltjes Integrals Generate Associated Normal RV's
  - Solutions are Readily Computed
  - Experimental Data will Guide Formulation of Functions
  - Ritz-Galerkin Methods will Resolve Equations of Motion
- Happy With the Results
- More to Come with Results of Computation



On Wiener Functionals

- Using Wiener Polynomial Functionals
  - Used HPF (Homogeneous Polynomial Functionals) for HIT
    - Used Perturbation Methods
    - Gratifying Results for Decay Laws
    - New Results for MFD Case
  - Used OPF (Orthogonal Polynomial Functionals) for PPF
    - Used Ritz-Galerkin Methods
    - General Agreement with Experiment
    - Continuing Work to Improve
- Problems
  - Convergence Issues at Distribution Tails
  - Hard Math and Computation

On Computation

- Using DNS Results
  - DNS Results (e.g. UT) Quickly lead to the $cdf^k(..)$ functions by simply sorting velocity values at the lattice points
  - The $q^k(..)$ come from applying the $cdf^k(..)$ to the velocity lattice
  - The $\phi^k(..)$ come from applying the $icdf^N(..)$ to the quantile lattice
  - The Autocorrelation lattice is obtained directly from $\phi^k(..)$ using standard tools
  - The $\Phi^k(..)$ are obtained by DFT from the Autocorrelation lattice
- Ritz-Galerkin Methods
  - Idea is to construct a Ritz-Galerkin Sequence of "Approximations"
  - Insert in NS equations to Yield a residual
  - Variance of Residual to be "Optimized" for "best fit"
- Other Matters

Conclusions - Again

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