Advanced Modeling & Simulation (AMS) Seminar Series
NASA Ames Research Center, March 11th, 2021

Turbulent scaling laws of wall-bounded shear flows for arbitrary moments, their symmetry basis, and its validation using DNS and experimental data

Martin Oberlack[1,2]
Sergio Hoyas[3]
Stefanie Kraheberger[1,2]
Francisco Alcántara-Ávila[3]
Jonathan Laux[1]
Dario Klingenberg[1,2]
Paul Hollmann[1]

Margit Vallikivi[4]
Marcus Hultmark[4]
Alexander Smits[4]

Gabriele Bellani[5]
Alessandro Talamelli[5]

Spencer Zimmerman[6]
Joseph Klewicki[6]

[1] Technische Universität Darmstadt, Germany
[2] Centre for Computational Engineering, TU Darmstadt, Germany
[3] Universitat Politècnica de València, Spain
[4] Princeton University, USA
[5] University of Bologna, Italy
[6] The University of Melbourne, Australia

MASCHINENBAU
We engineer future FDY

Funding:
MINECO/FEDER RTI2018-102256-B-I00, GVA/FEDER ACIF2018, DFG grant No. OB96/39-1, LRZ SuperMUC project pr92la.

Introduction

TECHNISCHE UNIVERSITÄT DARMSTADT

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

Model equation – Navier-Stokes

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

TECHNISCHE UNIVERSITÄT DARMSTADT

- Navier-Stokes equation – one of the 6 *unsolved* millennium problems of mathematics

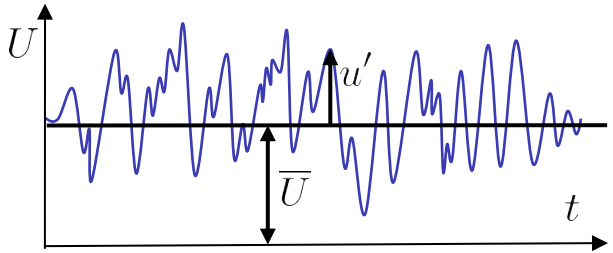
$$\frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{U} \cdot \nabla)\,\boldsymbol{U} = -\nabla P + \nu \Delta \boldsymbol{U}$$

$$\nabla \cdot \boldsymbol{U} = 0$$

To-date the “best model for turbulence”!

Turbulence closure problem

- Statistics of turbulence



- Reynolds decomposition

$$\overline{\boldsymbol{U}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \boldsymbol{U}\,\mathrm{d}t$$

$$\boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u'}$$

$$\frac{\partial \overline{\boldsymbol{U}}}{\partial t} + (\overline{\boldsymbol{U}} \cdot \nabla)\overline{\boldsymbol{U}} = -\nabla \overline{P} + \nu \Delta \overline{\boldsymbol{U}} - \nabla \cdot \boxed{\overline{\boldsymbol{u'} \otimes \boldsymbol{u'}}}$$

Unclosed Reynolds stress tensor

$\Rightarrow$ Empirical closure relations needed or considering all higher moments!

Turbulent scaling laws & log-law

- According to Prandtl / von Karman

$$\frac{\mathrm{d}\bar{U}}{\mathrm{d}y} = f(u_\tau, y, \cancel{\nu})$$

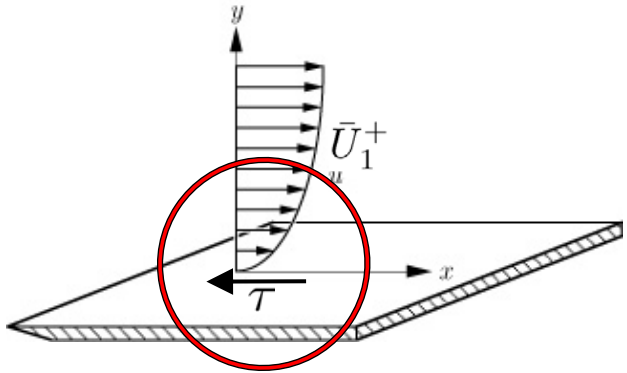
- with wall-friction velocity

$$u_\tau = \sqrt{\tau/\rho}$$

- Dimensional analysis

$$\frac{\mathrm{d}\bar{U}}{\mathrm{d}y} = \frac{1}{\kappa}\frac{u_\tau}{y}$$

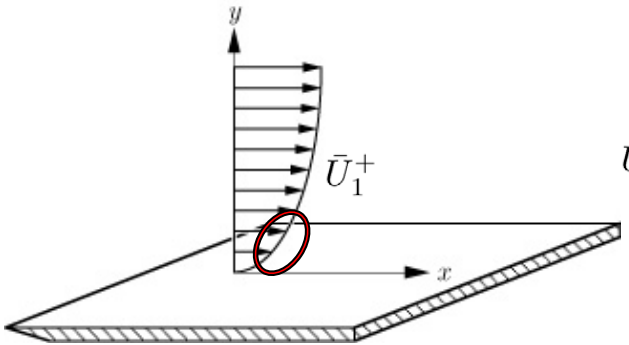
$$\Rightarrow \quad \boxed{\bar{U}_1^+ = \frac{1}{\kappa}\ln\left(x_2^+\right) + C}$$



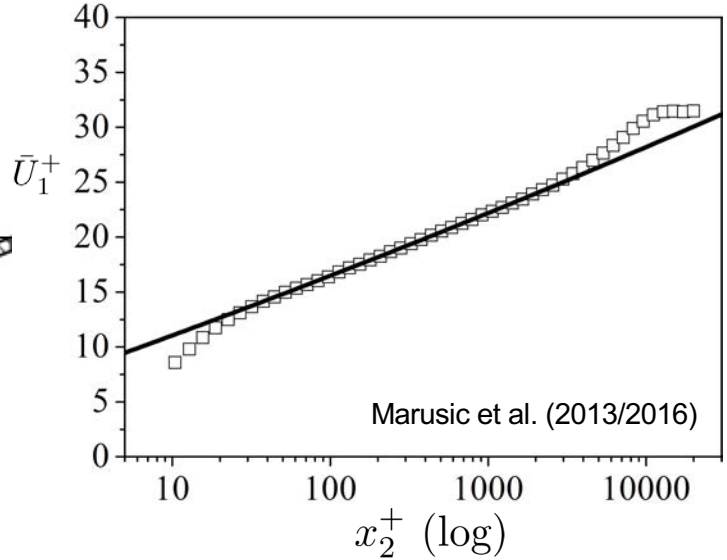
- with $y^+ = \dfrac{y u_\tau}{\nu}$

$$\bar{U}^+ = \frac{\bar{U}}{u_\tau}$$

The log-law of the wall



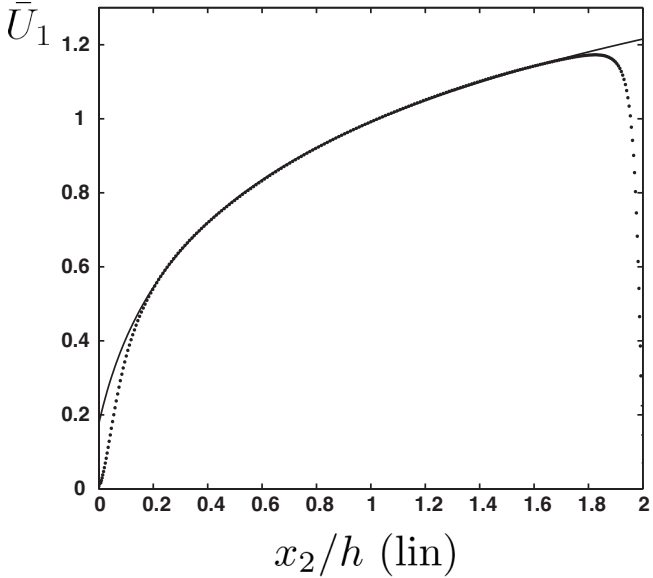
$$\bar{U}_1^+ = \frac{1}{\kappa} \ln\left(x_2^+ + A^+\right) + C$$

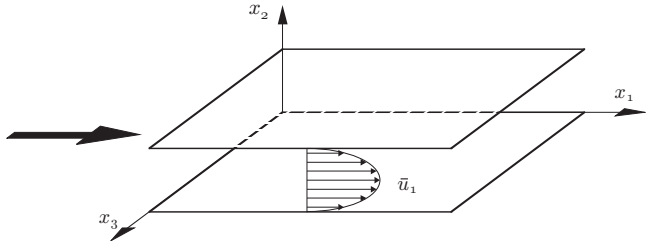


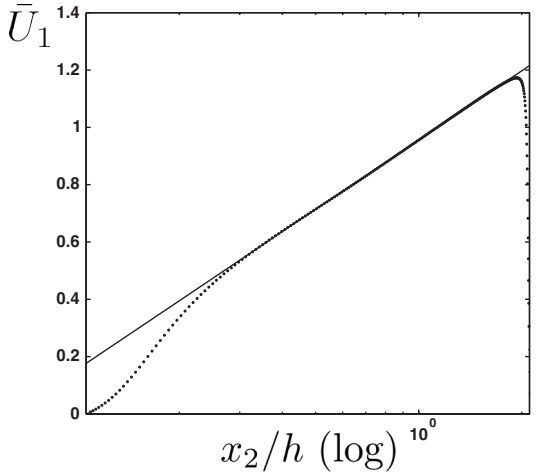
There's absolutely no connection to Navier-Stokes whatsoever!

Brainteaser: a new log-law?

- Why is there a log-law in the center?







Symmetries

TECHNISCHE UNIVERSITÄT DARMSTADT

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

Concept of symmetries

Analogy between symmetric object and differential equation

- Symmetric object (under rotation)



Symmetry of differential equations

- Differential equation $\mathcal{D}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(1)}, \dots) = 0$
- Symmetry transformation $\boldsymbol{x}^* = f(\boldsymbol{x}, \boldsymbol{u}) \quad , \quad \boldsymbol{u}^* = g(\boldsymbol{x}, \boldsymbol{u})$
- Form invariance $\mathcal{D}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}^{(1)}, \dots) = 0 \Leftrightarrow \mathcal{D}(\boldsymbol{x}^*, \boldsymbol{u}^*, \boldsymbol{u}^{*(1)}, \dots) = 0$

Symmetry of differential equation

Example: 1D heat equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

Symmetry transformation: scaling (two parameter)

$$S_{a_1,a_2}: \quad x^* = \mathrm{e}^{a_1} x \quad , \quad t^* = \mathrm{e}^{2a_1} t \quad , \quad T^* = \mathrm{e}^{a_2} T$$

into heat equation:

$$\Rightarrow \cancel{\mathrm{e}^{2a_1 - a_2}} \frac{\partial T^*}{\partial t^*} = \cancel{\mathrm{e}^{2a_1 - a_2}} \frac{\partial^2 T^*}{\partial x^{*2}}$$

Invariants

An invariant does not change its functional form under a given symmetry transformation

Example: Scaling of 1D heat equation

$$S_{a_1,a_2}: \quad x^* = \mathrm{e}^{a_1} x \quad , \quad t^* = \mathrm{e}^{2a_1} t \quad , \quad T^* = \mathrm{e}^{a_2} T$$

Invariants

1) $$\gamma = \frac{x}{\sqrt{t}} = \frac{\mathrm{e}^{-a_1} x^*}{\sqrt{\mathrm{e}^{-2a_1} t}} = \frac{x^*}{\sqrt{t^*}} = \gamma^* \qquad \left[\frac{\mathrm{d}x}{a_1 x} = \frac{\mathrm{d}t}{2a_1 t} \right]$$

2) $$\Delta = \frac{T}{t^{\frac{a_2}{2a_1}}} = \frac{\mathrm{e}^{-a_2} T^*}{\left(\mathrm{e}^{-2a_1} t^*\right)^{\frac{a_2}{2a_1}}} = \frac{T^*}{t^{*\frac{a_2}{2a_1}}} = \Delta^* \qquad \left[\frac{\mathrm{d}T}{a_2 T} = \frac{\mathrm{d}t}{2a_1 t} \right]$$

Invariant solutions

- *Implementing the invariants into the DE leads to a (similarity) reduction*
- Example: 1D heat equation

$$\tilde{x} = \frac{x}{\sqrt{t}}\,, \qquad \tilde{T} = \frac{T}{t^{\frac{a_2}{2a_1}}}$$

- Employed as independent and dependent variables into the heat equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \qquad \Rightarrow \qquad \frac{\mathrm{d}^2\tilde{T}}{\mathrm{d}\tilde{x}^2} + \frac{1}{2}\tilde{x}\frac{\mathrm{d}\tilde{T}}{\mathrm{d}\tilde{x}} - \frac{a_2}{2a_1}\tilde{T} = 0$$

Symmetry breaking

- *Specific (boundary) conditions may adjust certain parameters*
- Boundary condition

$$x = 0 : \; T = T_0$$

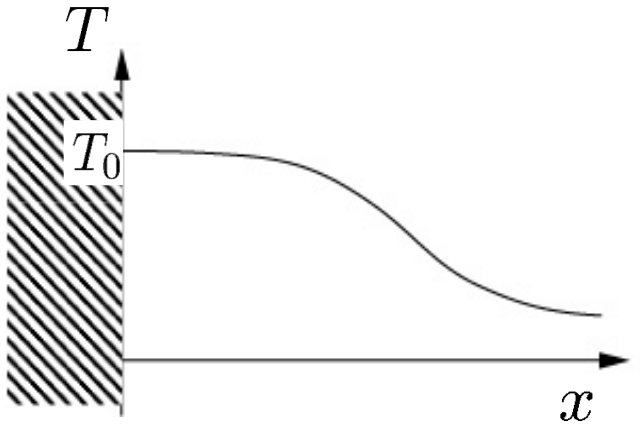
- Combined with the scaling symmetry

$$\Rightarrow \quad \mathrm{e}^{-a_1} x^* = 0 : \; \mathrm{e}^{-a_2} T^* = T_0$$

- Preserving the form of the BC

$$\Rightarrow \quad a_2 = 0 \,, \quad a_1$$ remains arbitrary

- T_0 is symmetry breaking



Symmetries of Navier-Stokes equations

Notation

$$\frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{U} \cdot \nabla)\, \boldsymbol{U} = -\nabla P + \nu \Delta \boldsymbol{U}$$

$$\nabla \cdot \boldsymbol{U} = 0$$

Symmetries of Navier-Stokes I

- Translation in time:

$$T_t : t^* = t + a_1 \ , \ \boldsymbol{x}^* = \boldsymbol{x} \ , \ \boldsymbol{U}^* = \boldsymbol{U} \ , \ P^* = P$$

- Finite rotation:

$$T_{r_1} - T_{r_3} : \ t^* = t \ , \ \boldsymbol{x}^* = \mathbf{a} \cdot \boldsymbol{x} \ , \ \boldsymbol{U}^* = \mathbf{a} \cdot \boldsymbol{U} \ , \ P^* = P$$

Note:

- $\mathbf{a} \neq \boldsymbol{f}(t)$

  (Flows are not invariant when sitting on a roundabout)

Symmetries of Navier-Stokes II

- Scaling of space:

$$T_{s_1} : t^* = t \ , \ \boldsymbol{x}^* = \mathrm{e}^{a_2}\boldsymbol{x} \ , \ \boldsymbol{U}^* = \mathrm{e}^{a_2}\boldsymbol{U} \ , \ P^* = \mathrm{e}^{2a_2}P$$

- Scaling of time:

$$T_{s_2} : t^* = \mathrm{e}^{a_3}t \ , \ \boldsymbol{x}^* = \boldsymbol{x} \ , \ \boldsymbol{U}^* = \mathrm{e}^{-a_3}\boldsymbol{U} \ , \ P^* = \mathrm{e}^{-2a_3}P$$

in the limit $\dfrac{1}{Re} \to 0$

Symmetries of Navier-Stokes III

- Galilean invariance:

$$T_{u_1} - T_{u_3} : t^* = t,\ \boldsymbol{x}^* = \boldsymbol{x} + \boldsymbol{f}(t),\ \boldsymbol{U}^* = \boldsymbol{U} + \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}t},\ P^* = P - \boldsymbol{x} \cdot \frac{\mathrm{d}^2\boldsymbol{f}}{\mathrm{d}t^2}$$

- Comprises two classical cases:
  - Translation in space: $\boldsymbol{f}(t) = \boldsymbol{a}$
  - Classical Galilean invariance: $\boldsymbol{f}(t) = \boldsymbol{b}\,t$
- A few more ... not relevant here ...

All symmetries transfer to statistical equations!

Turbulence and Symmetries

TECHNISCHE UNIVERSITÄT DARMSTADT

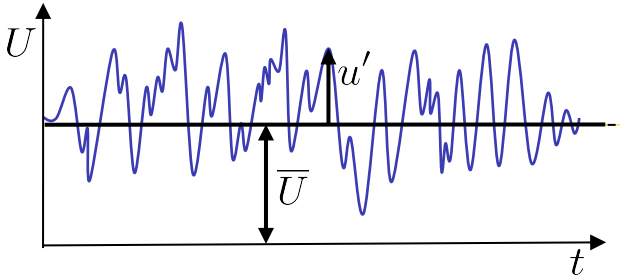
UNIVERSITAT POLITÈCNICA DE VALÈNCIA

Statistics of turbulence

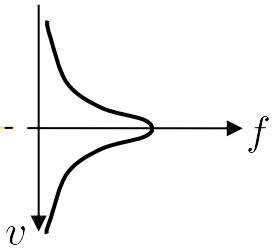
UNIVERSITAT POLITÈCNICA DE VALÈNCIA

TECHNISCHE UNIVERSITÄT DARMSTADT

- Randomness of turbulence



- Probability density function (PDF)



- Time average

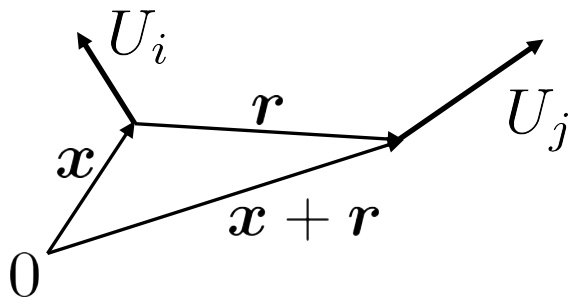
$$\overline{\boldsymbol{U}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \boldsymbol{U}\,\mathrm{d}t$$

- 1. Moment

$$\overline{\boldsymbol{U}} = \int \boldsymbol{v} f(\boldsymbol{v}, \boldsymbol{x})\,\mathrm{d}\boldsymbol{v}$$

Notations in turbulence

- Instantaneous value $\boldsymbol{U}, \quad P$
- Mean value $\bar{\boldsymbol{U}}, \quad \bar{P}$
- Fluctuating quantities $\boldsymbol{u}, \quad p$



- Two-point correlation tensor (classical notation)

$$R_{ij}\left(\boldsymbol{x},\boldsymbol{r},t\right)=\overline{u_i\left(\boldsymbol{x},t\right)u_j\left(\boldsymbol{x}+\boldsymbol{r},t\right)}$$

- New definition based on instantaneous velocities

$$H_{ij}\left(\boldsymbol{x},\boldsymbol{r},t\right)=\overline{U_i\left(\boldsymbol{x},t\right)U_j\left(\boldsymbol{x}+\boldsymbol{r},t\right)}$$

Statistical moments in turbulence

- 1-point moment (mean velocity)

$$H_i\left(\boldsymbol{x}\right) = \overline{U}_i\left(\boldsymbol{x}\right) = \int \boldsymbol{v} f(\boldsymbol{v}, \boldsymbol{x}) \mathrm{d}\boldsymbol{v}$$

- 2-point moment

$$H_{ij}\left(\boldsymbol{x}_{(1)}, \boldsymbol{x}_{(2)}\right) = \overline{U_i\left(\boldsymbol{x}_{(1)}\right) U_j\left(\boldsymbol{x}_{(2)}\right)} = \int \boldsymbol{v}_{(1)} \boldsymbol{v}_{(1)} f_2 \mathrm{d}\boldsymbol{v}_{(1)} \mathrm{d}\boldsymbol{v}_{(1)}$$

- n-point moment

$$H_{i_{\{n+1\}}} = H_{i_{(0)} i_{(1)} \ldots i_{(n)}} = \overline{U_{i_{(0)}}\left(\boldsymbol{x}_{(0)}\right) \cdot \ldots \cdot U_{i_{(n)}}\left(\boldsymbol{x}_{(n)}\right)} = \ldots$$

Relations

- Non-linear relations among tensor $R_{i_{\{n+1\}}}$, $\bar{U}_i$ and $H_{i_{\{n+1\}}}$

$$
\begin{aligned}
H_{i_{(0)}} &= \bar{U}_{i_{(0)}} \\
H_{i_{(0)}i_{(1)}} &= \bar{U}_{i_{(0)}}\bar{U}_{i_{(1)}} + R_{i_{(0)}i_{(1)}} \\
H_{i_{(0)}i_{(1)}i_{(2)}} &= \bar{U}_{i_{(0)}}\bar{U}_{i_{(1)}}\bar{U}_{i_{(2)}} \\
&\quad + R_{i_{(0)}i_{(1)}}\bar{U}_{i_{(2)}} + R_{i_{(0)}i_{(2)}}\bar{U}_{i_{(1)}} + R_{i_{(1)}i_{(2)}}\bar{U}_{i_{(0)}} + R_{i_{(0)}i_{(1)}i_{(2)}} \\
&\vdots \qquad\qquad \vdots
\end{aligned}
$$

- Unique relation between $R_{i_{\{n+1\}}}$, $\bar{U}_i$ and $H_{i_{\{n+1\}}}$ $\Rightarrow$ we use $H_{i_{\{n+1\}}}$!

Multi-point moment equation

- Navier-Stokes equation → Moment equation

$$\frac{\partial U_i}{\partial t} + \frac{\partial U_i U_k}{\partial x_k} = -\frac{\partial P}{\partial x_i} + \nu \Delta U_i \quad \Big| \cdot \; U_{i_{(0)}} U_{i_{(1)}} \dots U_{i_{(n)}}$$

$$\Rightarrow \boxed{\frac{\partial H_{i_{\{n+1\}}}}{\partial t} + \sum_{l=0}^{n} \left[\frac{\partial H_{i_{\{n+2\}}[i_{(n+1)} \mapsto k_{(l)}]} \left[\boldsymbol{x}_{(n+1)} \mapsto \boldsymbol{x}_{(l)} \right]}{\partial x_{k_{(l)}}} + \frac{\partial I_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}} - \nu \frac{\partial^2 H_{i_{\{n+1\}}}}{\partial x_{k_{(l)}} \partial x_{k_{(l)}}} \right] = 0}$$

$$\text{for } n = 1, \dots, \infty$$

- **Constitutes an <u>infinite</u> set of <u>linear</u> PDEs**

Multi-point correlation: Fluctuation approach (classical)

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

TECHNISCHE UNIVERSITÄT DARMSTADT

- Definition multi-point tensor

$$R_{i_{\{n+1\}}} = R_{i_{(0)} i_{(1)} \ldots i_{(n)}} = \overline{u_{i_{(0)}}(\boldsymbol{x}_{(0)}) \cdot \ldots \cdot u_{i_{(n)}}(\boldsymbol{x}_{(n)})}$$

$$\Rightarrow \frac{\partial R_{i_{\{n+1\}}}}{\partial t} + \sum_{l=0}^{n} \left[\bar{U}_{k_{(l)}}(\boldsymbol{x}_{(l)}) \frac{\partial R_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}} + R_{i_{\{n+1\}}[i_{(l)} \mapsto k_{(l)}]} \frac{\partial \bar{U}_{i_{(l)}}(\boldsymbol{x}_{(l)})}{\partial x_{k_{(l)}}} \right.$$
$$+ \frac{\partial P_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}} - \nu \frac{\partial^2 R_{i_{\{n+1\}}}}{\partial x_{k_{(l)}} \partial x_{k_{(l)}}} - R_{i_{\{n\}}[i_{(l)} \mapsto \emptyset]} \frac{\partial \overline{u_{i_{(l)}} u_{k_{(l)}}}(\boldsymbol{x}_{(l)})}{\partial x_{k_{(l)}}}$$
$$\left. + \frac{\partial R_{i_{\{n+2\}}[i_{(n+1)} \mapsto k_{(l)}][\boldsymbol{x}_{(n+1)} \mapsto \boldsymbol{x}_{(l)}]}}{\partial x_{k_{(l)}}} \right] = 0$$

- Infinite set of non-linear PDEs; coupling among all orders

New statistical symmetry I

$$\frac{\partial H_{i_{\{n+1\}}}}{\partial t}+\sum_{l=0}^{n}\left[\frac{\partial H_{i_{\{n+2\}}[i_{(n+1)}\mapsto k_{(l)}]}[\boldsymbol{x}_{(n+1)}\mapsto\boldsymbol{x}_{(l)}]}{\partial x_{k_{(l)}}}+\frac{\partial I_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}}-\nu\frac{\partial^2 H_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}\partial x_{k_{(l)}}}\right]=0$$

- Translation in function space:

$$\bar{T}'_{2_{\{n\}}}:\quad t^*=t,\quad \boldsymbol{x}^*=\boldsymbol{x},\quad \boldsymbol{r}^*_{(l)}=\boldsymbol{r}_{(l)},\quad \mathbf{H}^*_{\{n\}}=\mathbf{H}_{\{n\}}+\mathbf{C}_{\{n\}},\quad \mathbf{I}^*_{\{n\}}=\mathbf{I}_{\{n\}}+\mathbf{D}_{\{n\}},$$

with $\mathbf{C}_{\{n\}}$ and $\mathbf{D}_{\{n\}}$ arbitrary constant tensors

- Key ingredient for log-law, etc. and turbulence models
- **Defines a measure of non-gaussianity**

New statistical symmetry II

$$\frac{\partial H_{i_{\{n+1\}}}}{\partial t}+\sum_{l=0}^{n}\left[\frac{\partial H_{i_{\{n+2\}}[i_{(n+1)}\mapsto k_{(l)}]}[\boldsymbol{x}_{(n+1)}\mapsto\boldsymbol{x}_{(l)}]}{\partial x_{k_{(l)}}}+\frac{\partial I_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}}-\nu\frac{\partial^2 H_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}\partial x_{k_{(l)}}}\right]=0$$

- Scaling of moments:

$$\bar{T}'_s:\quad t^*=t,\quad \boldsymbol{x}^*=\boldsymbol{x},\quad \boldsymbol{r}^*_{(l)}=\boldsymbol{r}_{(l)},\qquad \mathbf{H}^*_{\{n\}}=\mathrm{e}^{a_s}\mathbf{H}_{\{n\}},\quad \mathbf{I}^*_{\{n\}}=\mathrm{e}^{a_s}\mathbf{I}_{\{n\}}$$

- **Defines a measure of intermittency**
- Important for deriving higher order moments
- Very difficult to implement into turbulence models

  $\Rightarrow$ *Klingenberg, Oberlack, Plümacher: Physics of Fluids, Phys. Fluids* ***32****, 025108 (2020)*

New statistical symmetry – PDF version

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

TECHNISCHE UNIVERSITÄT DARMSTADT

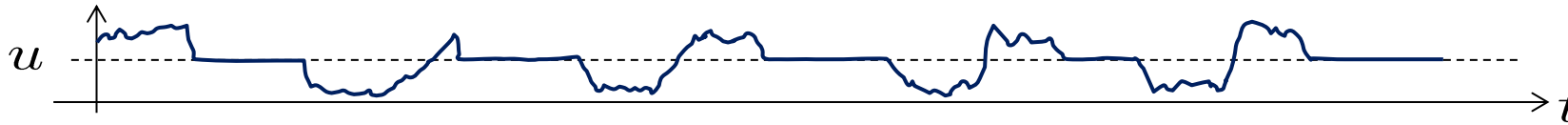
- Scaling of correlations/moments

$$\bar{T}'_s: \quad t^* = t, \quad \boldsymbol{x}^* = \boldsymbol{x}, \quad \boldsymbol{r}^*_{(l)} = \boldsymbol{r}_{(l)}, \quad \mathbf{H}^*_{\{n\}} = \mathrm{e}^{a_s}\mathbf{H}_{\{n\}}, \quad \mathbf{I}^*_{\{n\}} = \mathrm{e}^{a_s}\mathbf{I}_{\{n\}}$$

- Equivalent symmetry of PDF (LMN-eqn)

$$f^*_n = \delta(\boldsymbol{v}_{(1)}, ..., \boldsymbol{v}_{(n)}) + \mathrm{e}^{a_s}\left(f_n - \delta(\boldsymbol{v}_{(1)}, ..., \boldsymbol{v}_{(n)})\right)$$

- Consequences
  - Mathematically: since f_n and f^*_n are non-negative functions $\Rightarrow a_s \geq 0$ (semi-group)
  - Physically: measure of intermittency



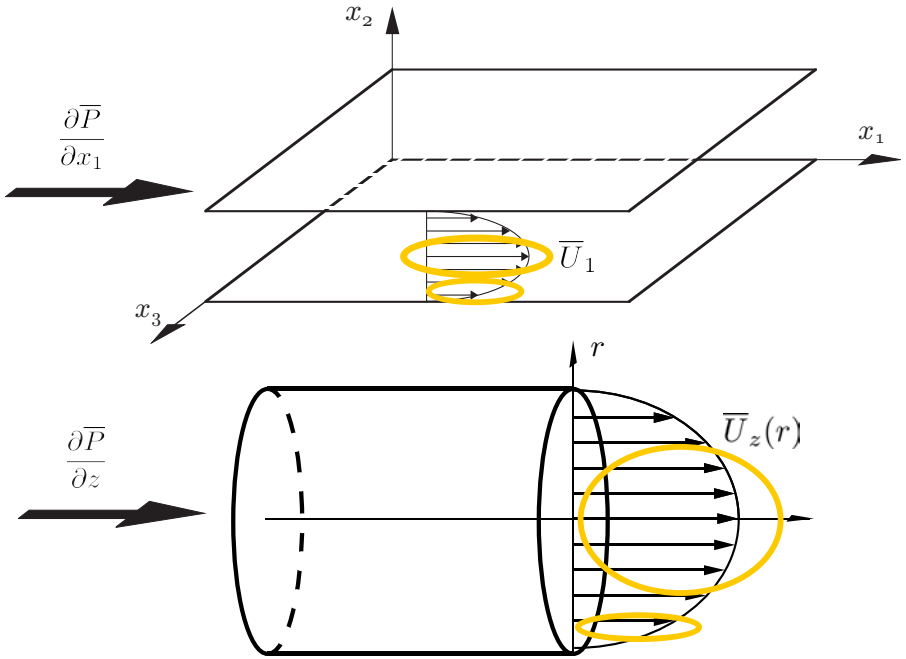
Symmetry induced scaling laws

TECHNISCHE UNIVERSITÄT DARMSTADT

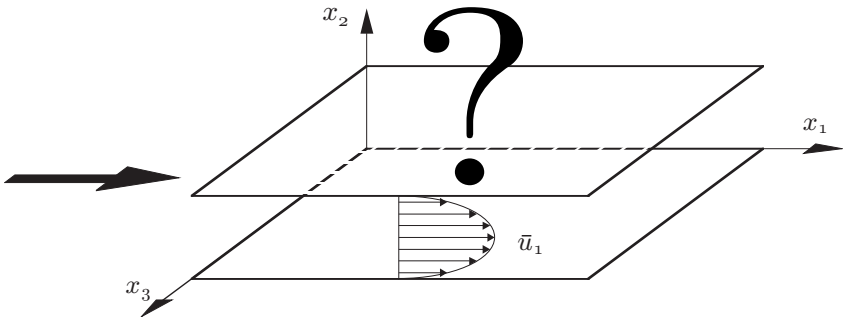
UNIVERSITAT POLITÈCNICA DE VALÈNCIA

Regions of turbulent scaling laws

- Channel flow and pipe scaling laws

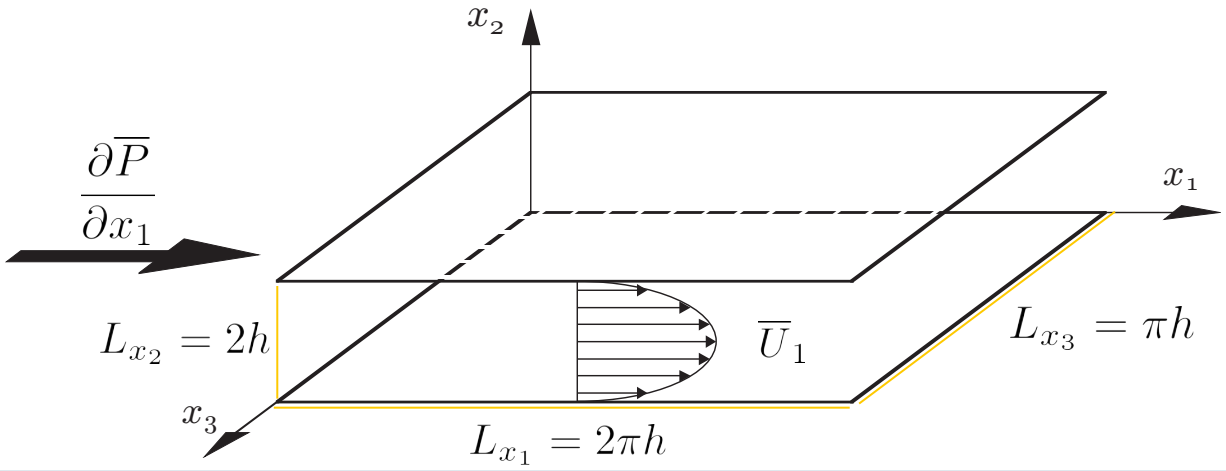


- Brain teaser - “modified“ channel flow



Channel flow DNS details I

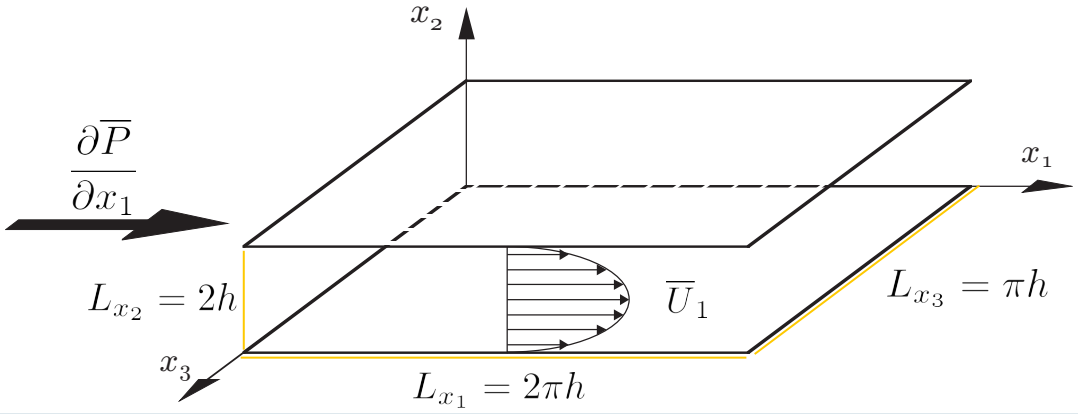
- Pure Poiseuille channel flow @ $\mathrm{Re}_\tau = 10^4$
- Conducted at SuperMuc, LRZ, Munich, Germany
- Code: LISO (Hoyas & Jimenez, 2006)
- Fourier in x_1 and x_3, compact finite differences in x_2, Runge-Kutta time-stepper



Channel flow DNS details II

Re_τ	Δx_1^+	Δx_3^+	$\Delta_{max} x_2^+$	L_{x_1}/h	L_{x_3}/h	$\Delta t u_\tau/h$	Ref.
2000	12.3	6.2	8.9	8π	3π	10.3	HJ06
4200	12.8	6.4	10.7	2π	π	15.0	LJ14
5200	12.7	6.4	10.3	8π	3π	7.8	LM15
10000	15.3	7.6	13.0	2π	π	14.10	

~ 60M CPU h



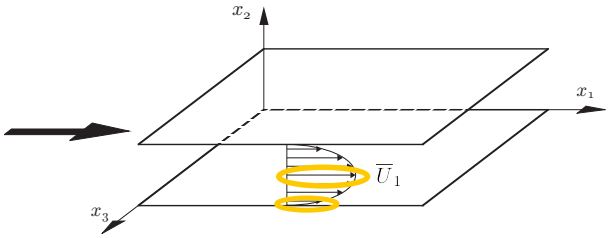
Pipe flows experiments

UNIVERSITAT POLITÈCNICA DE VALÈNCIA
TECHNISCHE UNIVERSITÄT DARMSTADT

- Two different experimental facilities:
  1. *Centre for International Cooperation in Long Pipe Experiments* (*CICLoPE*), Bologna
  2. *SuperPipe* facility of the *Gas Dynamics Lab* at Princeton University
- Large range of friction Reynolds numbers: Re_τ = *2000 - 94000*

Facility	Reynolds numbers	Sensors	Autors
CICLoPE 1	5200, 7700, 10000, 14000, 35000	Multi-Element Hot-Wire	Spencer Zimmerman Joseph Klewicki
CICLoPE 2	9900, 22000, 32000, 35000	Hot-Wire	Gabriele Bellani Alessandro Talamelli
SuperPipe	2000, 3000, 5400, 10000, 20000, 38000, 68000, 94000	NSTAP (Hot-Wire)	Margit Vallikivi Marcus Hultmark Alexander Smits

Specific symmetries for shear flows



- Fundamental geometrical assumption for the flow

$$\bar{\boldsymbol{U}} = [\bar{U}_1(x_2), 0, 0] \ , \quad H_{ij} = H_{ij}(x_2) \ , \ \cdots$$

- Related symmetries:
  - Scaling:

$$\bar{T}_{s_1-s_3} : (t^* = \mathrm{e}^{a_3}\, t), \ x_2^* = \mathrm{e}^{a_2}\, x_2, \ \bar{U}_1^* = \mathrm{e}^{a_2-a_3+a_s}\, \bar{U}_1,$$

$$H_{ij}^* = \mathrm{e}^{2(a_2-a_3)+a_s} H_{ij} \ , \ \ldots \ , \ H_{i_{\{n\}}}^* = \mathrm{e}^{n(a_2-a_3)+a_s} H_{i_{\{n\}}}$$

  - Translation in function space:

$$\bar{T}_{\bar{U}_1} : \ x_2^* = x_2 \ , \ \bar{U}_1^* = \bar{U}_1 + a_1 \ , \ H_{ij}^* = H_{ij} + a_{i_{\{n\}}} \ , \ldots$$

  - Translational invariance in x_2 – direction:

$$\bar{T}_{x_2} : \ x_2^* = x_2 + a_4 \ , \ \bar{U}_1^* = \bar{U}_1 \ , \ H_{ij}^* = H_{ij} \ , \ldots$$

Wall-bounded parallel shear flows

UNIVERSITAT POLITÈCNICA DE VALÈNCIA
TECHNISCHE UNIVERSITÄT DARMSTADT

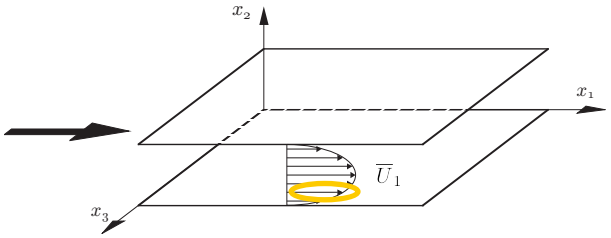
- Invariance condition / scaling law equations

$$\frac{\mathrm{d}\bar{U}_1}{\mathrm{d}x_2} = \frac{(a_2 - a_3 + a_s)\bar{U}_1 + a_1}{a_2 x_2 + a_4}, \quad \cdots, \quad \frac{\mathrm{d}H_{i_{\{n\}}}}{\mathrm{d}x_2} = \frac{[n(a_2 - a_3) + a_s]H_{i_{\{n\}}} + a_{i_{\{n\}}}}{a_2 x_2 + a_4}$$

- $a_s, a_1, \ldots, a_{i_{\{n\}}}$ are the group parameter of the statistical symmetries
- Recall *Prandtl / von Karman* equation

$$\frac{\mathrm{d}\bar{U}}{\mathrm{d}y} = \frac{1}{\kappa}\frac{u_\tau}{y}$$

Log-region – 1st moment / mean velocity



- Symmetry breaking: wall-friction velocity $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$

$$\Rightarrow a_2 - a_3 + a_s = 0$$

- Scaling invariance

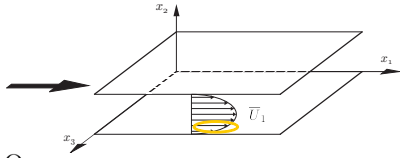
$$x_2^* = \mathrm{e}^{a_2}\, x_2 \;, \quad t^* = \mathrm{e}^{a_3}\, t \;, \quad \bar{U}_1^* = \mathrm{e}^{\overbrace{a_2 - a_3 + a_s}^{=0}}\, \bar{U}_1 \;, \quad \dots$$

- Solution

$$\Rightarrow \quad \frac{\mathrm{d}\bar{U}_1}{\mathrm{d}x_2} = \frac{\overbrace{(a_2 - a_3 + a_s)}^{=0}\,\bar{U}_1 + a_1}{a_2 x_2 + a_4}$$

$$\Rightarrow \quad \boxed{\bar{U}_1^+ = \tfrac{1}{\kappa} \ln\left(x_2^+ + A^+\right) + B}$$

Log-region – higher moments



- Invariance condition for higher moments in the log-region

$$\frac{\mathrm{d}H_{i_{\{n\}}}}{\mathrm{d}x_2} = \frac{[n(a_2 - a_3) + a_s]H_{i_{\{n\}}} + a_{i_{\{n\}}}}{a_2 x_2 + a_4} \quad \wedge \quad a_2 - a_3 + a_s = 0$$

- Invariant solution for the higher moments for $n \geq 2$

Contains constant of integration c_n

$$H^+_{1_{\{n\}}} = C_n \left(x_2^+ + A^+\right)^{\omega(n-1)} - B_n$$

with

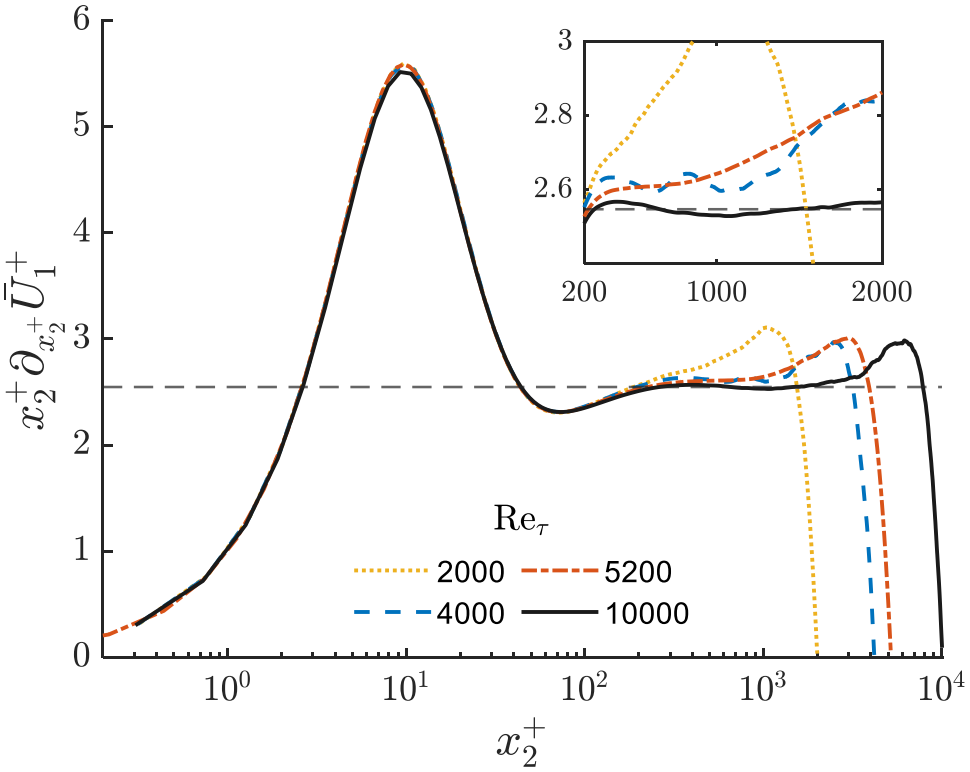
$$C_n = \alpha \mathrm{e}^{\beta n},\ B_n = \tilde{\alpha} \mathrm{e}^{\tilde{\beta} n}$$

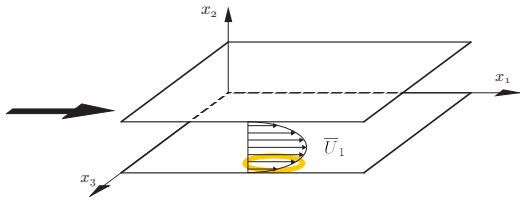
- Note: ω is the exponent of the second moment $\Rightarrow$ determines all higher moments

  ω and κ emerge form a combination of classical and statistical symmetries

  Scaling is not due to “dimensional reasoning” $\Rightarrow$ Intermittency induced scaling!

DNS: Log-region – mean velocity scaling

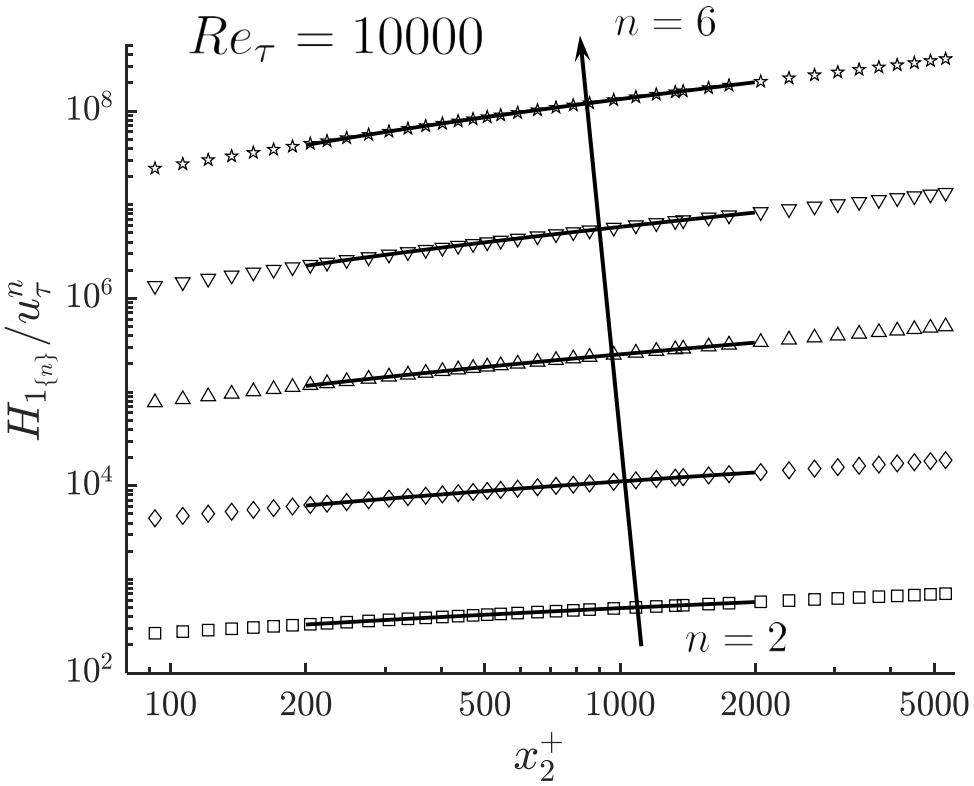


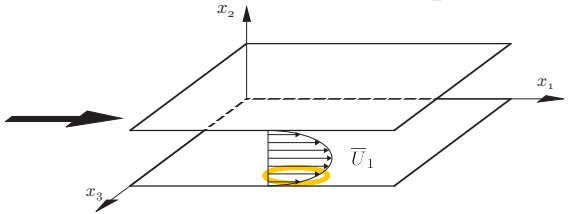


Re_τ	κ	B
1000	0.39	4.51
2000	0.37	3.85
4000	0.38	4.27
5200	0.38	4.26
10000	0.39	4.44

$$\overline{U}_1^+ = \frac{1}{\kappa} \ln\left(x_2^+ + A\right) + B$$

DNS: Log-region – higher moments scaling

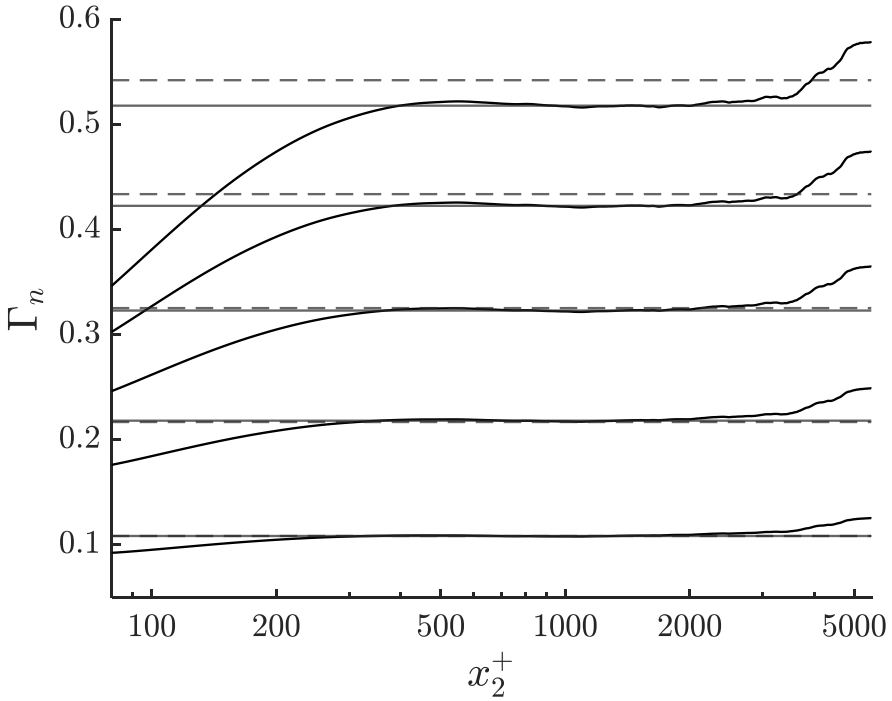


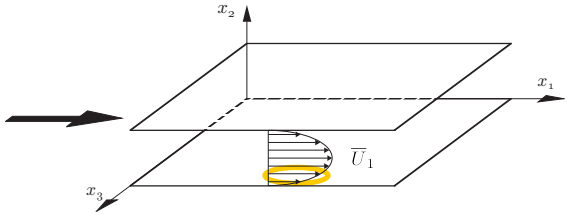


n	DNS	$\omega(n-1)$
2	0.11	-
3	0.22	0.22
4	0.32	0.33
5	0.42	0.43
6	0.52	0.54

$$H^+_{1\{n\}} = C_n \left(x_2^+ + A^+\right)^{\omega(n-1)} - B_n$$

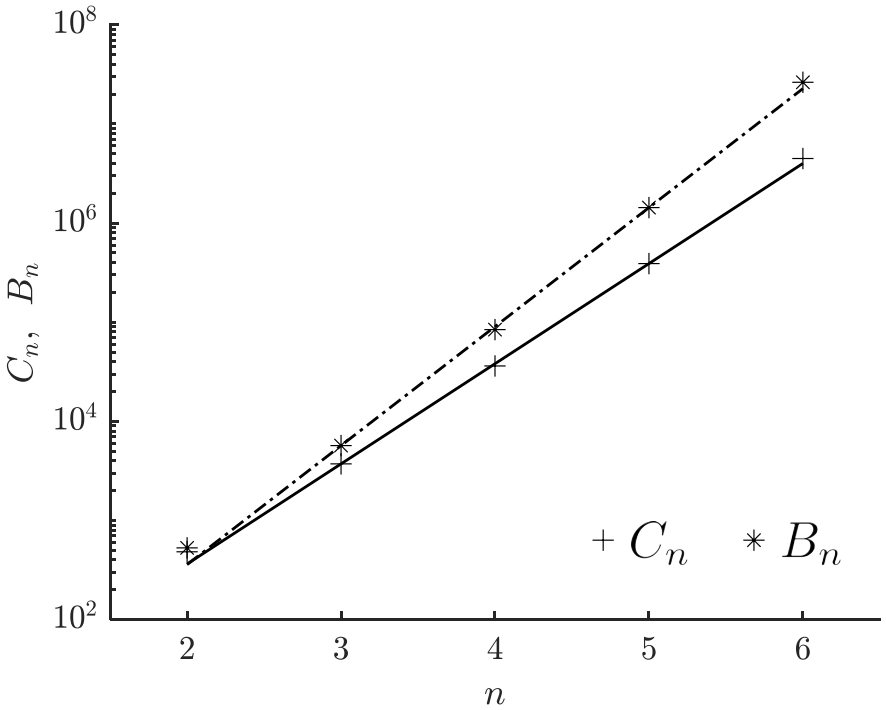
DNS: Log-region – moments indicator funct.

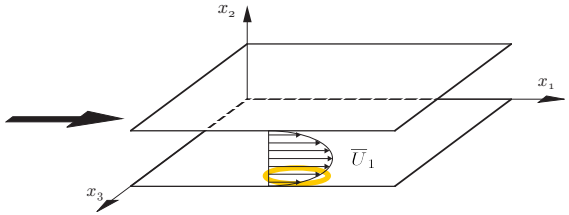




$$\Gamma_n = \frac{x_2^+}{H_{1\{n\}}^+ + B_n} \frac{\mathrm{d}H_{1\{n\}}^+}{\mathrm{d}x_2^+} \stackrel{!}{=} \omega(n-1)$$

DNS: Log-region – prefct. scaling of C_n, B_n





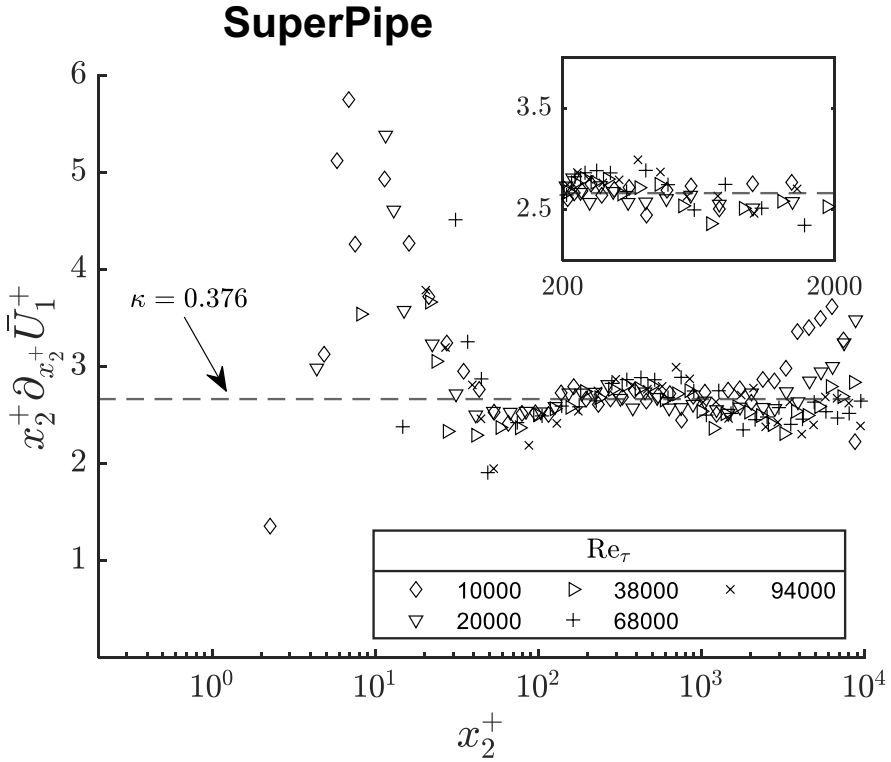
$$H^+_{1_{\{n\}}} = C_n \left(x_2^+ + A^+\right)^{\omega(n-1)} - B_n$$

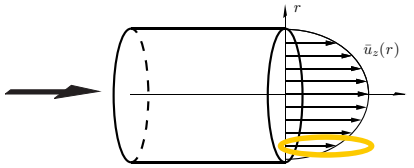
$$C_n = \alpha \mathrm{e}^{\beta n},\ B_n = \tilde{\alpha} \mathrm{e}^{\tilde{\beta} n}$$

$$\alpha = 3.48 \qquad \tilde{\alpha} = 1.43$$

$$\beta = 2.33 \qquad \tilde{\beta} = 2.76$$

EXP: Log-region – mean velocity scaling





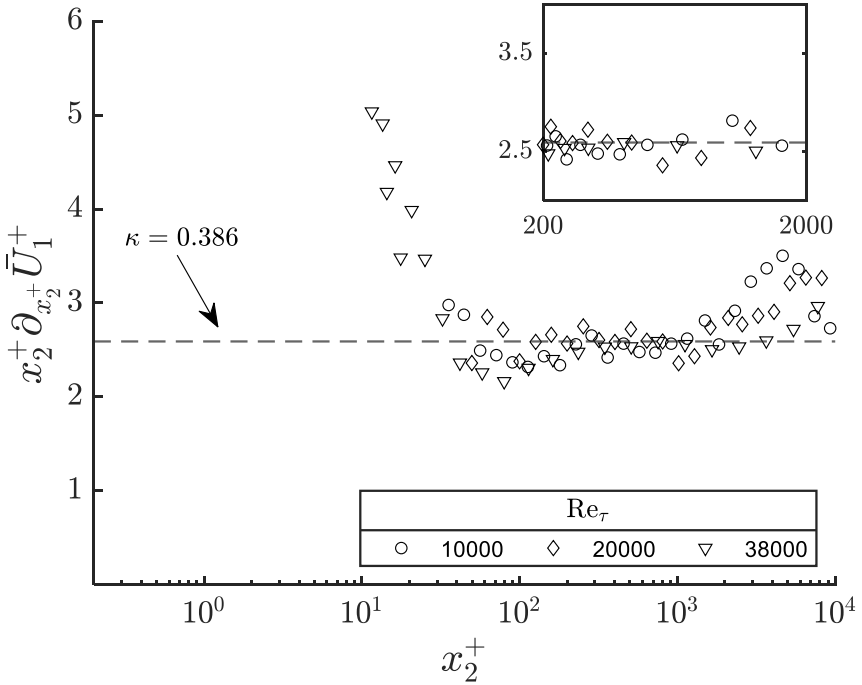
$$\overline{U}_1^+ = \frac{1}{\kappa} \ln\left(x_2^+ + A\right) + B$$

EXP: Log-region – mean velocity scaling

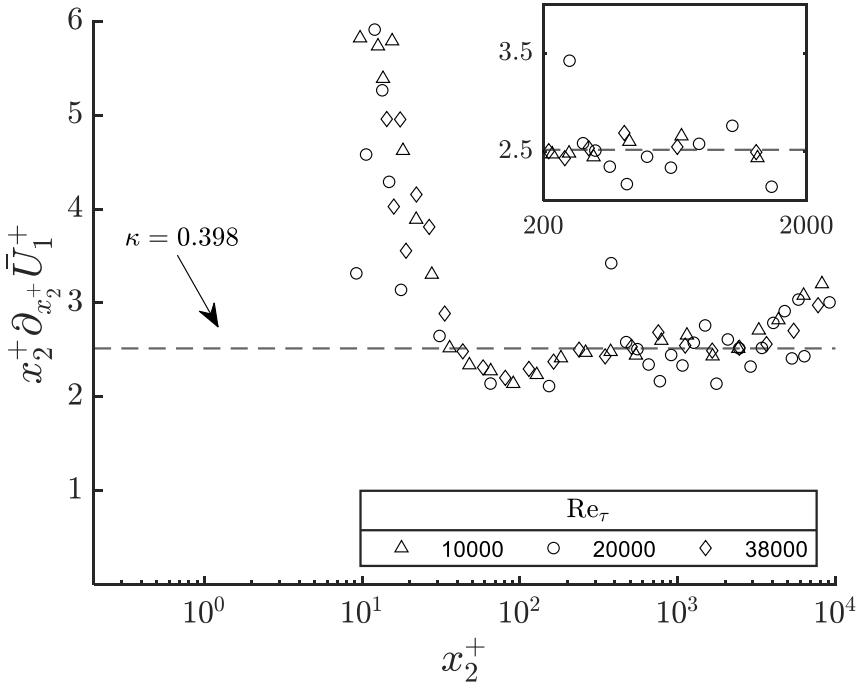
UNIVERSITAT POLITÈCNICA DE VALÈNCIA

TECHNISCHE UNIVERSITÄT DARMSTADT

CICLoPE 1



CICLoPE 2

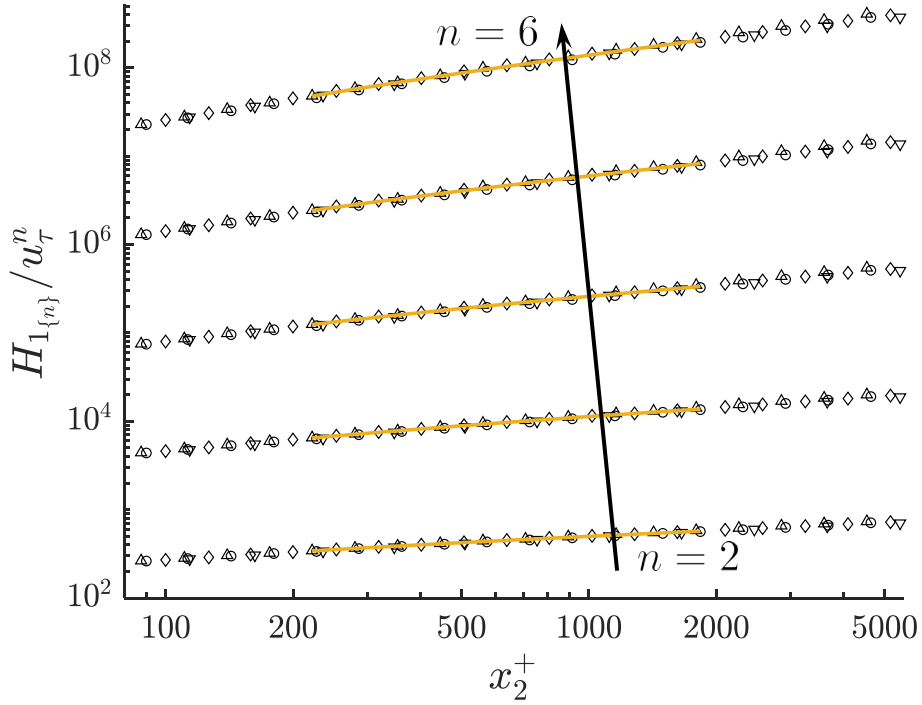


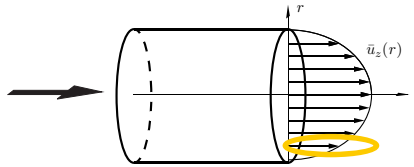
EXP: Log-region – higher moments scaling

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

TECHNISCHE UNIVERSITÄT DARMSTADT

CICLoPE 1

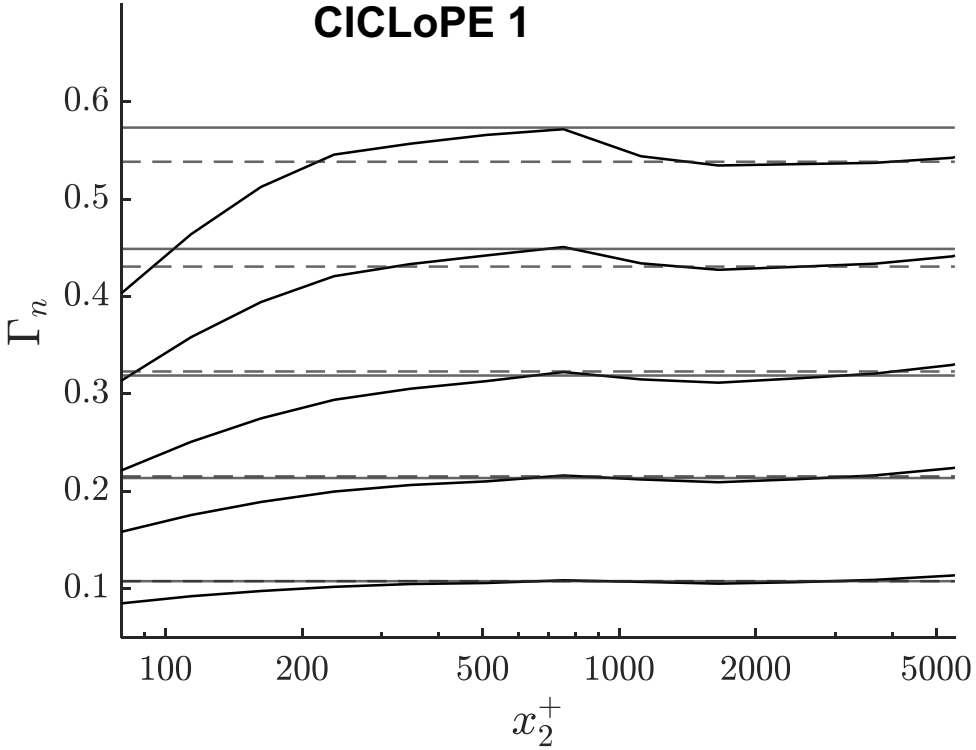


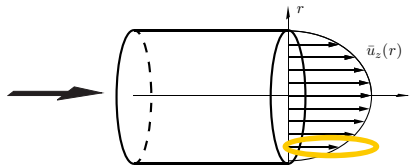


n	EXP	$\omega(n-1)$
2	0.11	-
3	0.21	0.22
4	0.32	0.32
5	0.45	0.43
6	0.57	0.54

$$H^+_{1_{\{n\}}} = C_n \left(x_2^+ + A^+\right)^{\omega(n-1)} - B_n$$

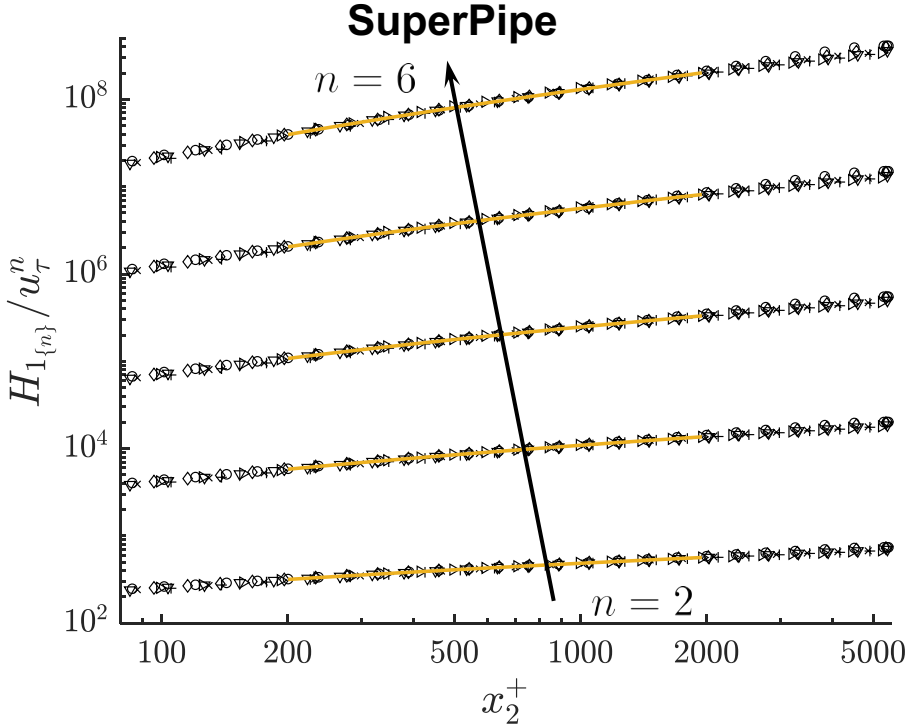
EXP: Log-region – moments indicator funct.

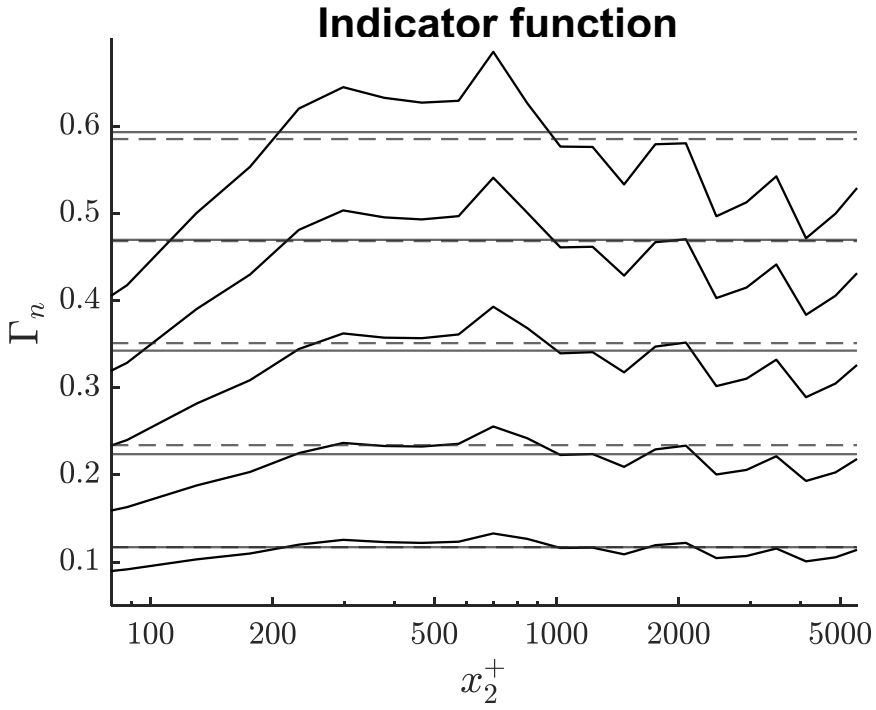




$$\Gamma_n = \frac{x_2^+}{H_{1_{\{n\}}}^+ + B_n} \frac{\mathrm{d}H_{1_{\{n\}}}^+}{\mathrm{d}x_2^+} \stackrel{!}{=} \omega(n-1)$$

EXP: Log-region – moments indicator funct.

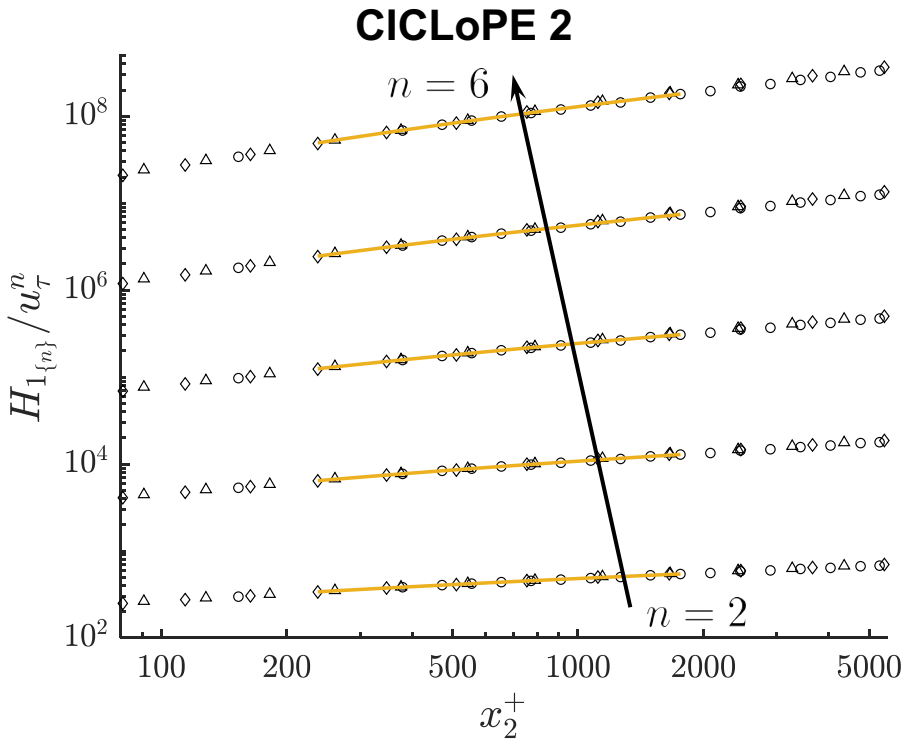


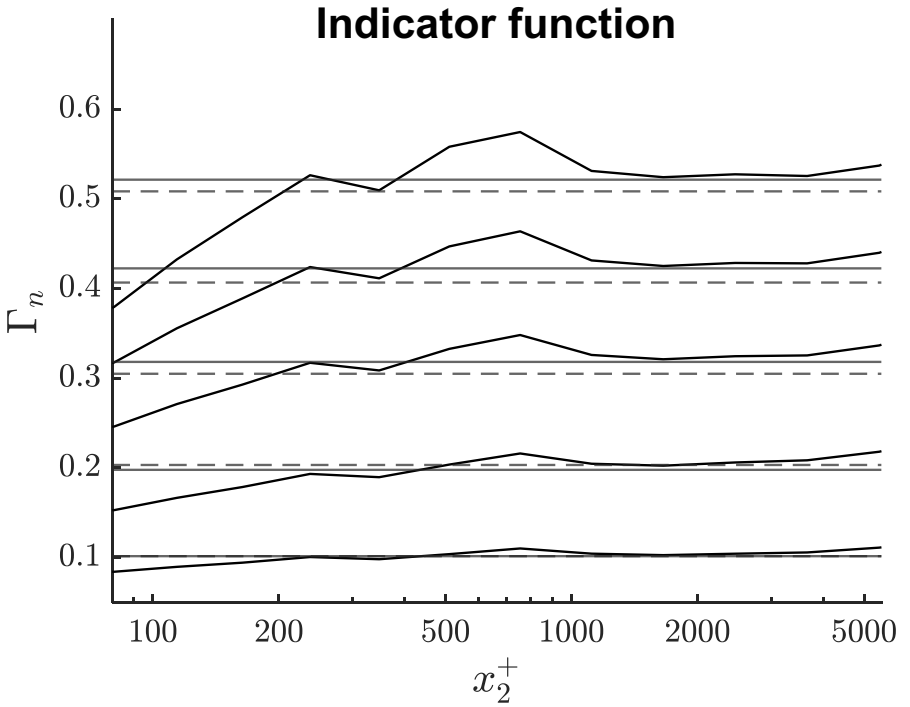


EXP: Log-region – moments indicator funct.

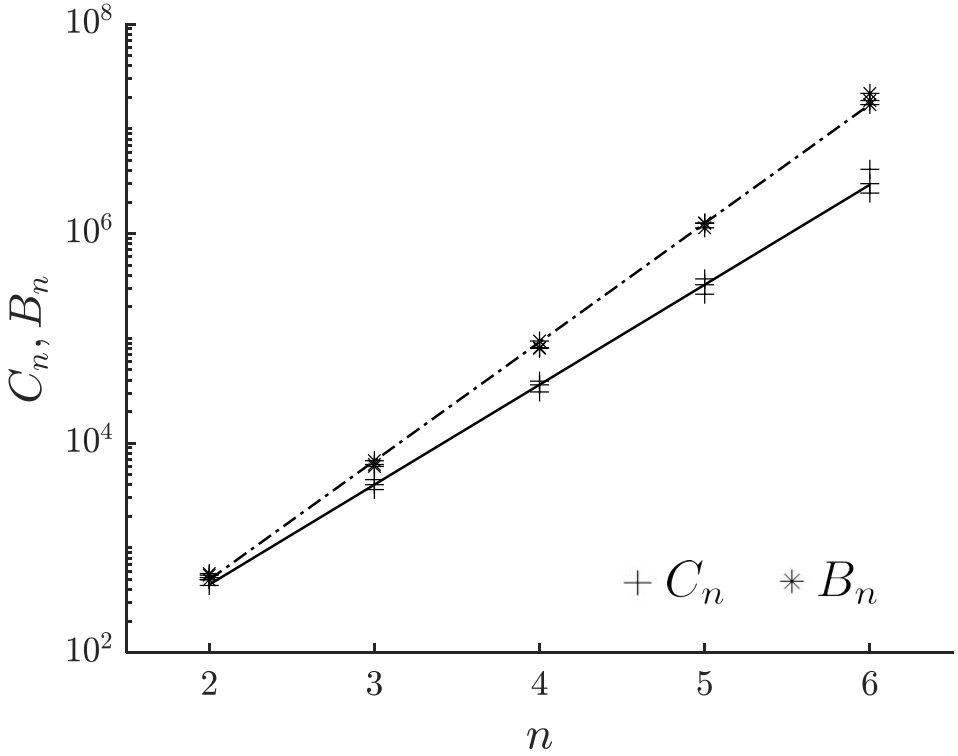
UNIVERSITAT POLITÈCNICA DE VALÈNCIA

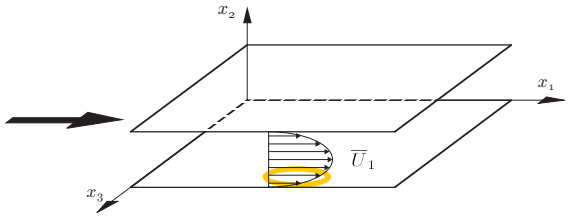
TECHNISCHE UNIVERSITÄT DARMSTADT





EXP: Log-region – prefct. scaling of C_n, B_n





$$H^+_{1_{\{n\}}} = C_n \left(x_2^+ + A^+\right)^{\omega(n-1)} - B_n$$

$$C_n = \alpha \mathrm{e}^{\beta n},\ B_n = \tilde{\alpha} \mathrm{e}^{\tilde{\beta} n}$$

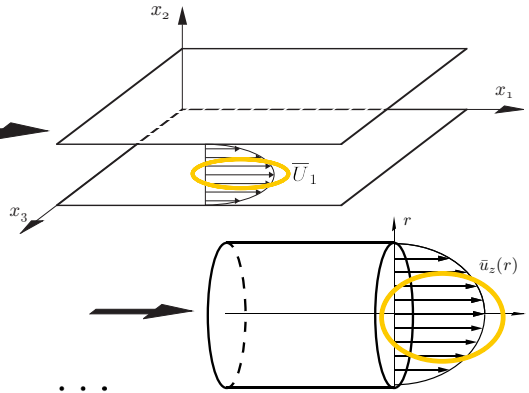
$$\alpha = 5.50, \quad \beta = 2.20$$

$$\tilde{\alpha} = 2.68, \quad \tilde{\beta} = 2.61$$

Centre / deficit region

- Symmetry breaking: unclear - most likely $\dfrac{u_\tau}{R^2} = \sqrt{\dfrac{\partial \bar{P}}{\partial x_1}\dfrac{1}{\rho R^3}}$

$$\Rightarrow a_2 - a_3 + a_s \neq 0$$



- Scale invariance

$$x_2^* = \mathrm{e}^{a_2}\, x_2 \;, \quad t^* = \mathrm{e}^{a_3}\, t \;, \quad \bar{U}_1^* = \mathrm{e}^{\overbrace{a_2 - a_3 + a_s}^{\neq 0}}\, \bar{U}_1 \;, \quad \dots$$

- Solution

$$\Rightarrow \quad \frac{\mathrm{d}\bar{U}_1}{\mathrm{d}x_2} = \frac{\overbrace{(a_2 - a_3 + a_s)}^{\neq 0}\,\bar{U}_1 + a_1}{a_2 x_2 + a_4}$$

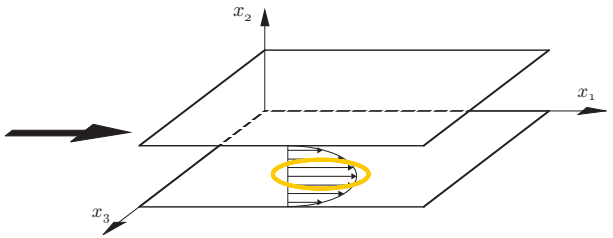
$$\Rightarrow \frac{\mathrm{d}H_{i_{\{n\}}}}{\mathrm{d}x_2} = \frac{[n(a_2 - a_3) + a_s]H_{i_{\{n\}}} + a_{i_{\{n\}}}}{a_2 x_2 + a_4}$$

Centre / deficit region scaling laws

- 1st moment / mean velocity

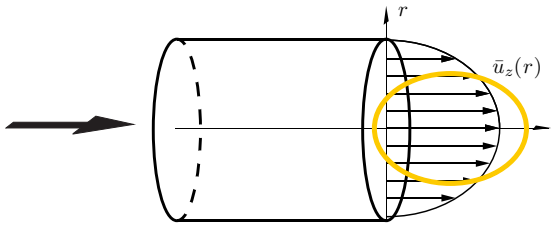
$$\frac{\bar{U}_{cl} - \bar{U}_1}{u_\tau} = C_1' \left(\frac{x_2}{h}\right)^{\sigma_1}$$

recall: $H_{1_{\{1\}}} = \bar{U}_1$



- 2nd moment

$$\frac{H_{1_{\{2\}}cl} - H_{1_{\{2\}}}}{u_\tau^2} = C_2' \left(\frac{x_2}{h}\right)^{\sigma_2}$$

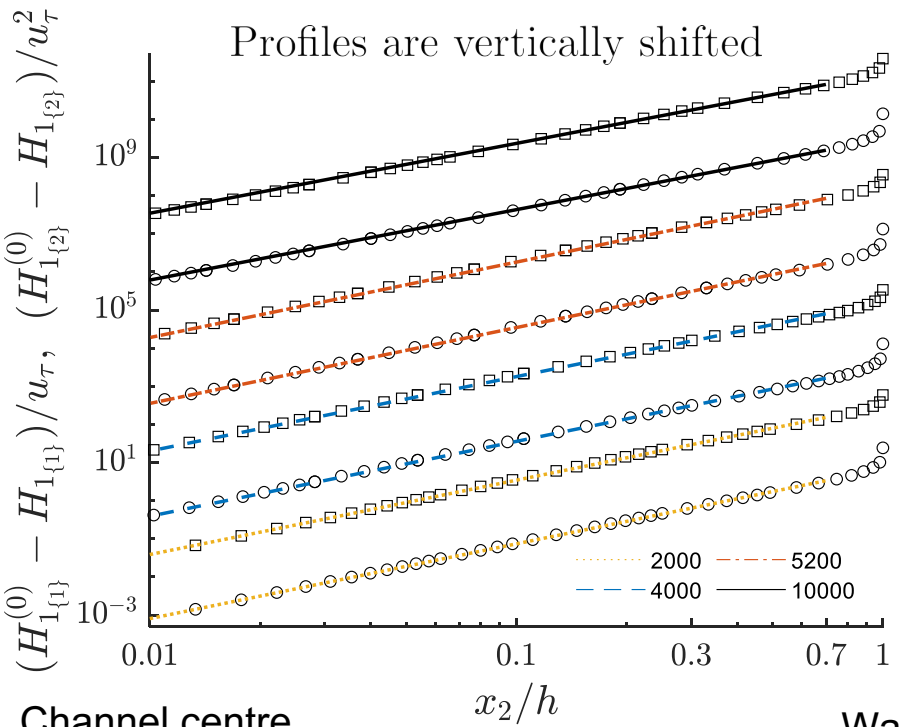


- Moments for $n > 2$

$$\frac{H_{1_{\{n\}}cl} - H_{1_{\{n\}}}}{u_\tau^n} = C_n' \left(\frac{x_2}{h}\right)^{n(\sigma_2-\sigma_1)+2\sigma_1-\sigma_2}$$

with $$C_n' = \alpha' \mathrm{e}^{\beta' n}$$

DNS: Centre region 1st & 2nd moment scaling

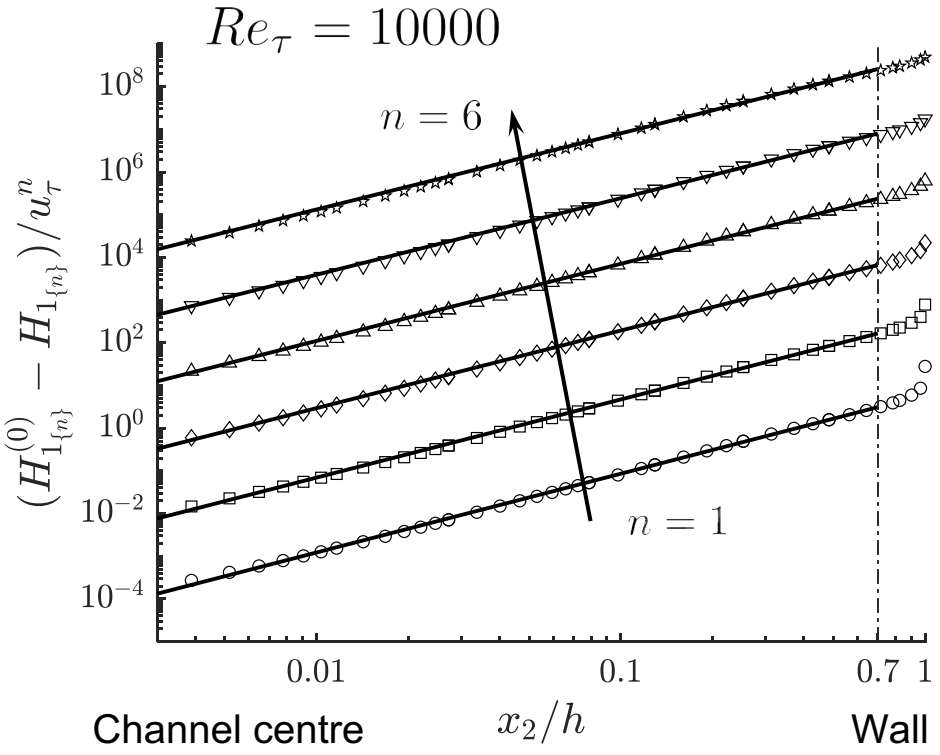


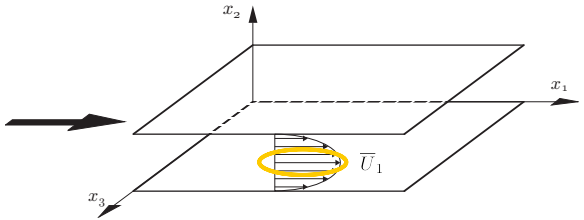
$$\frac{\bar{U}_{cl} - \bar{U}_1}{u_\tau} = C_1' \left(\frac{x_2}{h}\right)^{\sigma_1}$$

$$\frac{H_{1\{2\}cl} - H_{1\{2\}}}{u_\tau^2} = C_2' \left(\frac{x_2}{h}\right)^{\sigma_2}$$

Re_τ	C_1'	C_2'	σ_1	σ_2
2000	6.75	306.93	1.96	1.95
4000	6.58	320.38	1.96	1.95
5200	6.74	342.78	1.99	1.98
10000	6.00	316.81	1.85	1.83

DNS: Centre region nth moment scaling





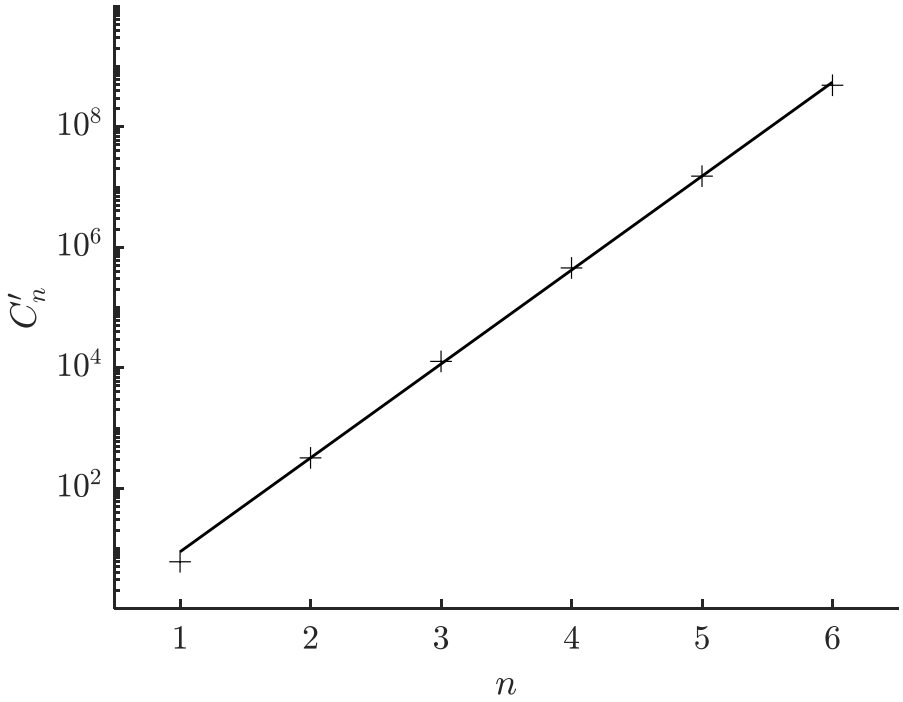
$$\frac{H_{1_{\{n\}}cl} - H_{1_{\{n\}}}}{u^n_\tau} = C'_n \left(\frac{x_2}{h}\right)^{n(\sigma_2-\sigma_1)+2\sigma_1-\sigma_2}$$

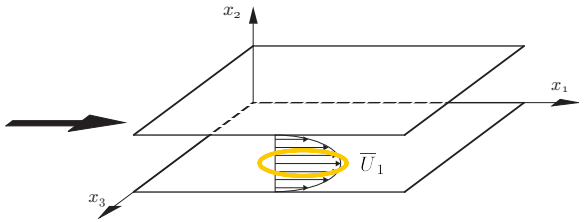
n	DNS	$n(\sigma_2-\sigma_1)$ $+2\sigma_1-\sigma_2$
1	1.84	-
2	1.83	-
3	1.81	1.82
4	1.81	1.80
5	1.79	1.79
6	1.78	1.77

Value almost zero

$\Rightarrow$ strongly intermittent!

DNS: Centre region – prefactor scaling of C_n'





$$\frac{H_{1_{\{n\}}cl} - H_{1_{\{n\}}}}{u_\tau^n} = C_n' \left(\frac{x_2}{h}\right)^{n(\sigma_2-\sigma_1)+2\sigma_1-\sigma_2}$$

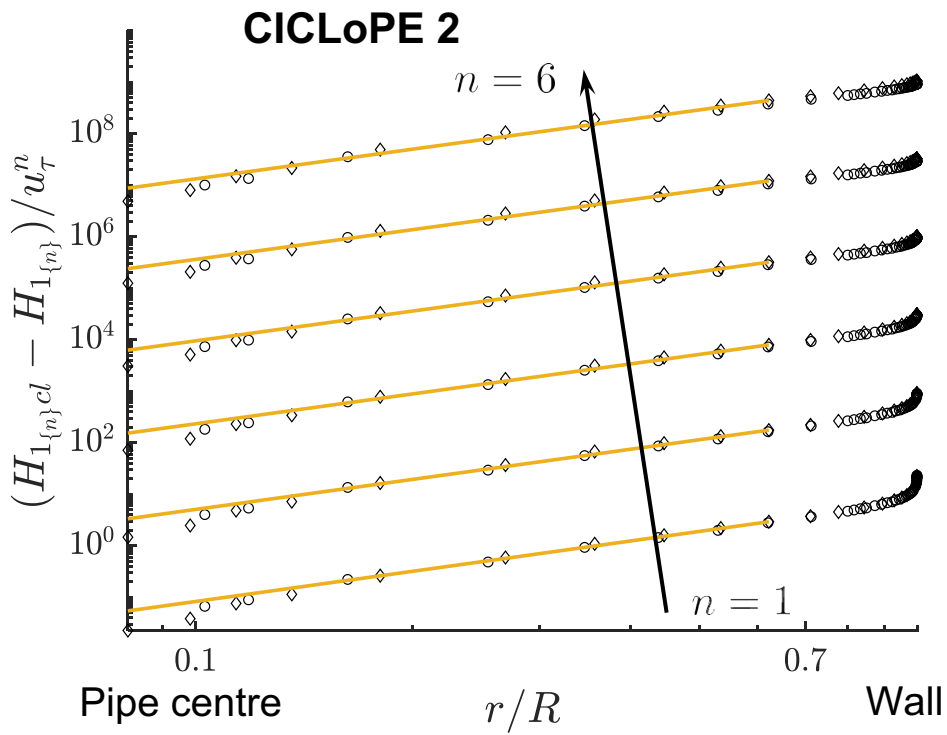
$$C_n' = \alpha' \mathrm{e}^{\beta' n}$$

$$\alpha' = 0.24 \qquad \beta' = 3.59$$

EXP: Centre region nth moment scaling

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

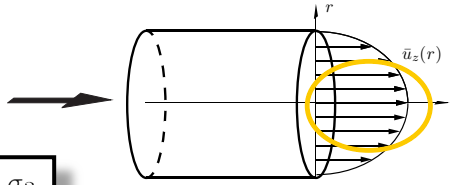
TECHNISCHE UNIVERSITÄT DARMSTADT



$$\frac{\bar{U}_{cl} - \bar{U}_1}{u_\tau} = d_{1_{\{1\}}} \left(\frac{r}{R}\right)^{\sigma_1}$$

$$\frac{H_{1_{\{2\}}cl} - H_{1_{\{2\}}}}{u_\tau^2} = d_{1_{\{2\}}} \left(\frac{r}{R}\right)^{\sigma_2}$$

$$\frac{H_{1_{\{n\}}cl} - H_{1_{\{n\}}}}{u_\tau^n} = d_{1_{\{n\}}} \left(\frac{r}{R}\right)^{n(\sigma_2-\sigma_1)+2\sigma_1-\sigma_2}$$

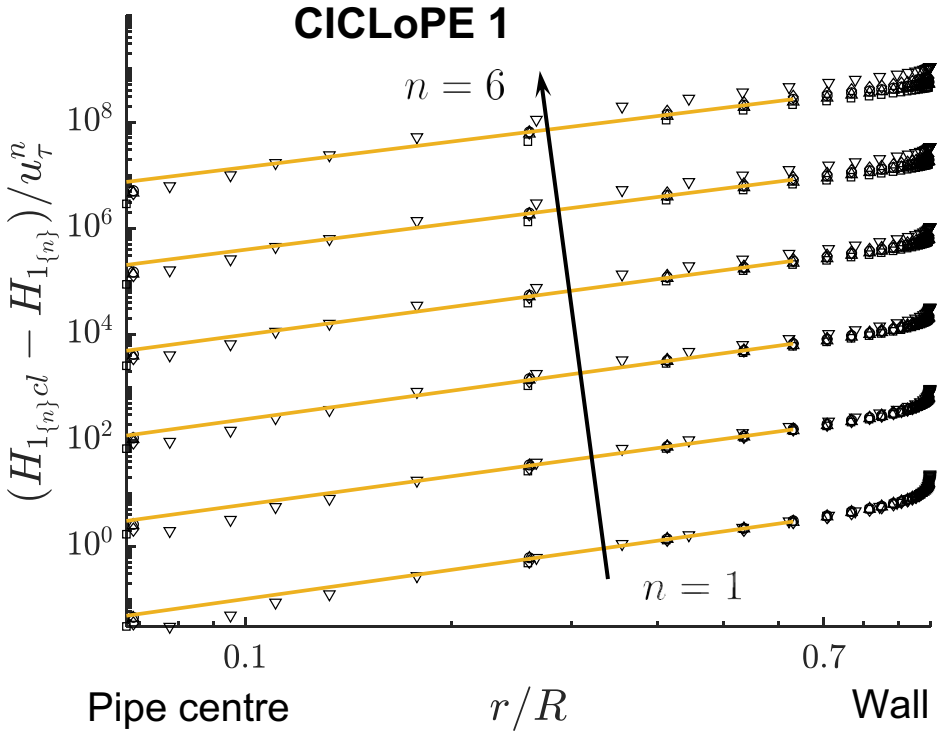


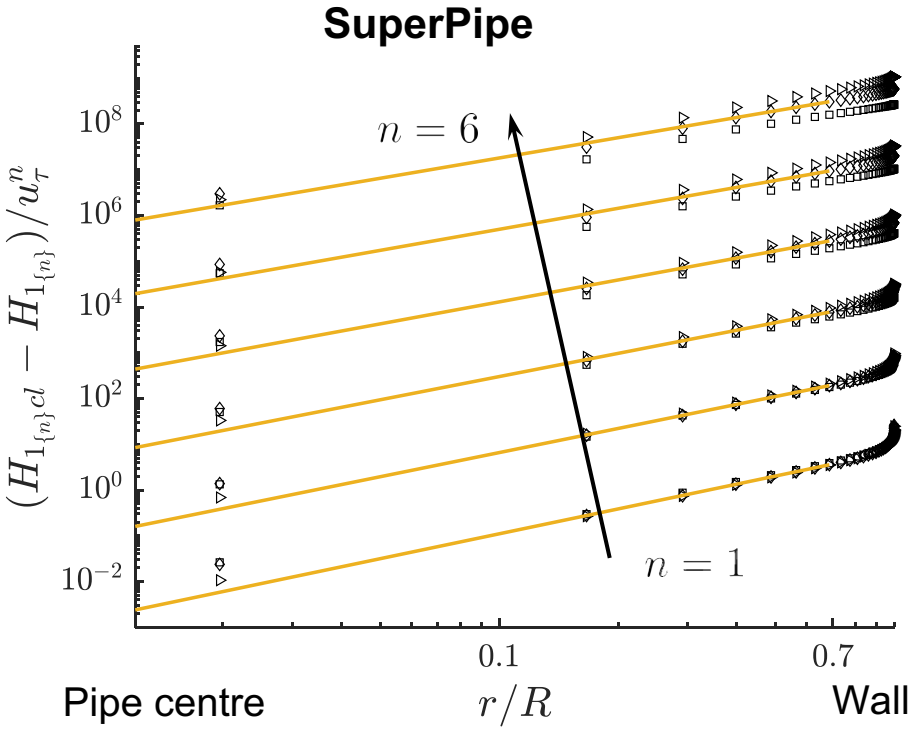
n	EXP	$n(\sigma_2-\sigma_1)$ $+2\sigma_1-\sigma_2$
1	1.95	-
2	1.94	-
3	1.93	1.93
4	1.92	1.92
5	1.92	1.91
6	1.92	1.89

EXP: Centre region nth moment scaling

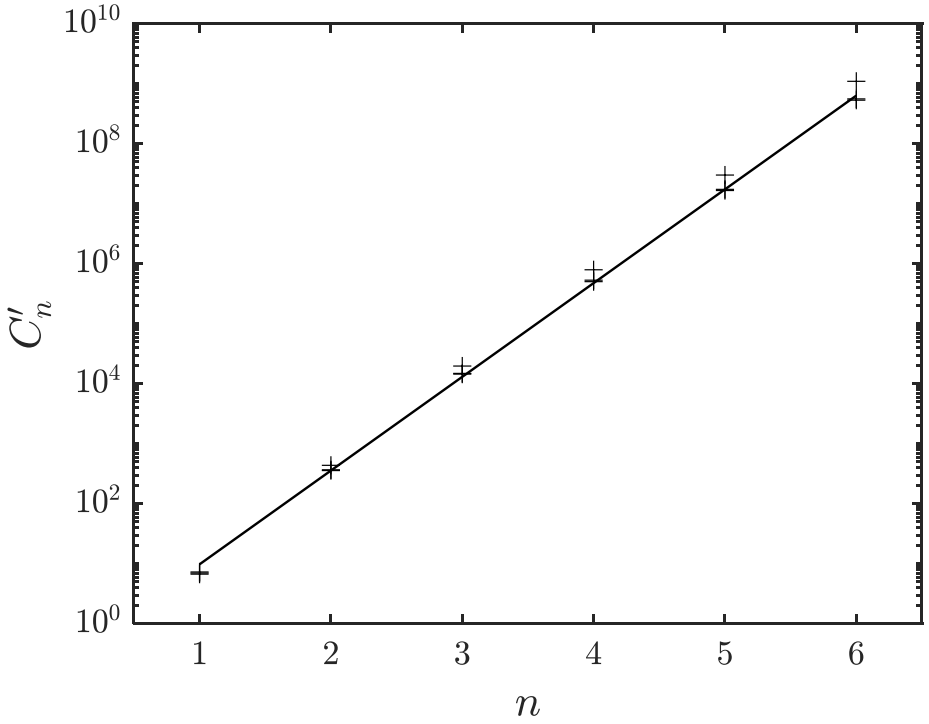
UNIVERSITAT POLITÈCNICA DE VALÈNCIA

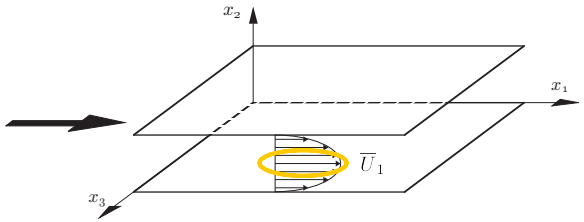
TECHNISCHE UNIVERSITÄT DARMSTADT





EXP: Centre region – prefactor scaling of C'_n



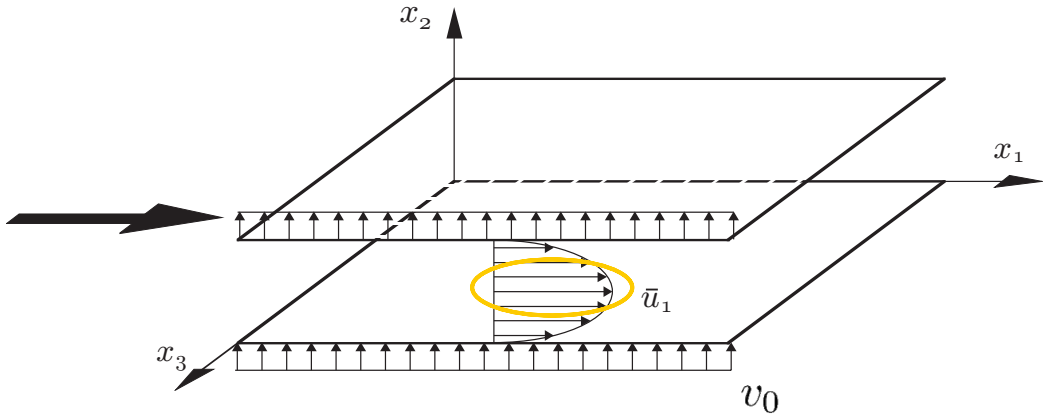


$$\frac{H_{1_{\{n\}}cl} - H_{1_{\{n\}}}}{u_\tau^n} = C'_n \left(\frac{r}{R}\right)^{n(\sigma_2-\sigma_1)+2\sigma_1-\sigma_2}$$

$$C'_n = \alpha' \mathrm{e}^{\beta' n}$$

$$\alpha' = 0.27, \quad \beta' = 3.60$$

Solution to the “brain teaser”: Channel flow with transpiration



- Symmetry suggested solution for the centre region

$$\bar{U}_1(x_2) = \hat{u} \ln \left[\alpha \left(\frac{x_2}{h} + \beta \right) + \gamma \right]$$

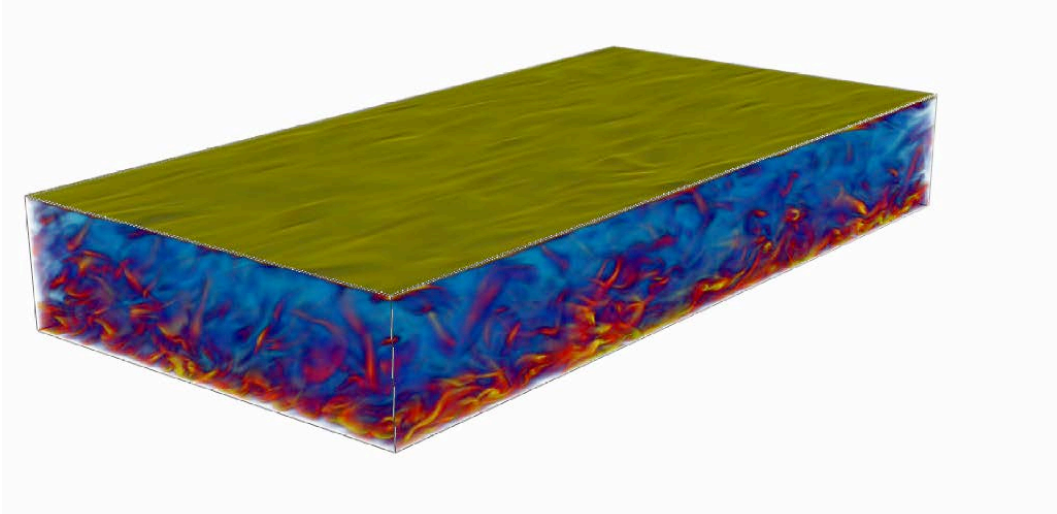
- Avsarkisov, Oberlack, Hoyas: J. Fluid Mech. (2014), 746

Channel flow with transpiration

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

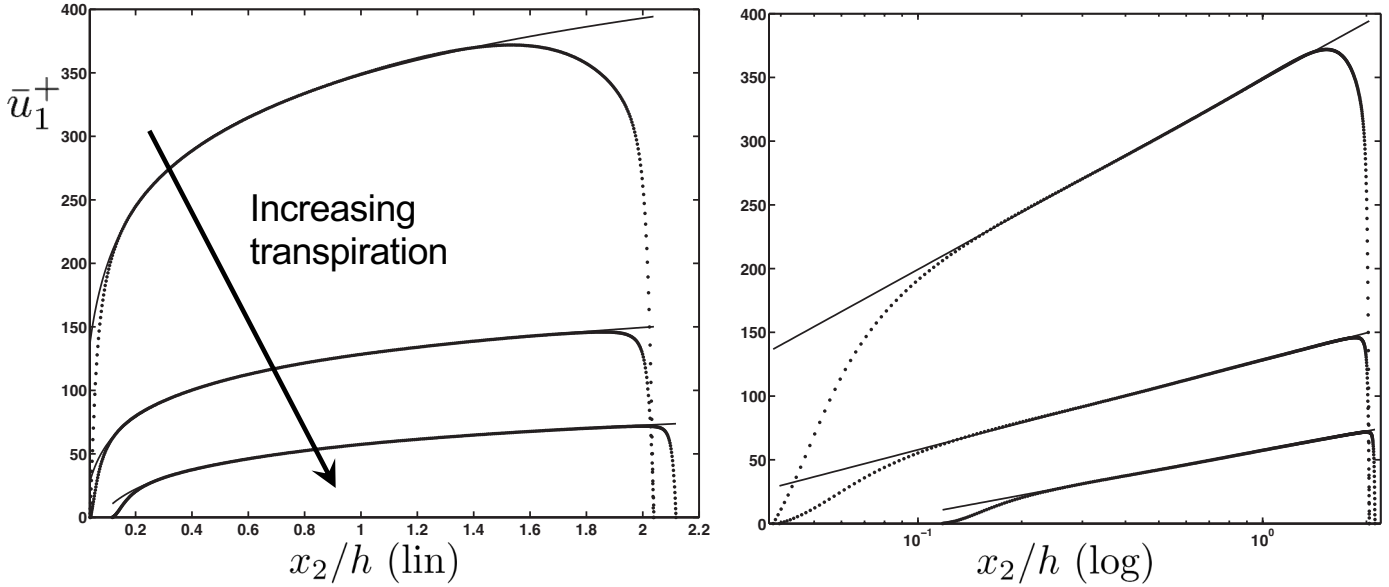
TECHNISCHE UNIVERSITÄT DARMSTADT

- Vorticity magnitude



- Avsakisov et al. J. Fluid Mech. (2014), vol. 746, pp. 99-122.

Channel flow with transpiration



$$\frac{\bar{U}_1(x_2) - U_{\text{bulk}}}{u_\tau} = \frac{1}{\gamma} \ln\left[\left(\frac{x_2}{h} + \beta\right)\right]$$

Conclusions

- New sets of statistical symmetries (beyond Navier-Stokes) have been derived
- Statistical symmetries mirror key properties of turbulence: *intermittency* and *non-gaussianity*
- They are the key ingredients for classical and new scaling laws here: wall turbulence, rotating, wall-transpiration, …
- In particular: moment scaling of arbitrary order have been derived

  ⇒ Scaling laws are solutions of the multi-point equations

  ⇒ Scaling laws can be derived rigorously from symmetries

  ⇒ Scaling laws are particularly simple in H-formulation

Thank you all for listening

It was a pleasure being here today

I am happy to take your questions

Note: all related publications may be taken from ResearchGate

MASCHINENBAU
We engineer future FDY

TECHNISCHE UNIVERSITÄT DARMSTADT

UNIVERSITAT POLITÈCNICA DE VALÈNCIA