Matrix Form of Finite Difference Schemes

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Non-Periodic Wave Equation

Example 1: Linear Wave Equation, non-periodic, 11 Grid Points

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$BC : u(0,t) = u_{left}$$

$$IC : u(x,0) = g(x)$$

Second Order Central Differencing

$$\frac{\partial u_j}{\partial t} + \frac{a}{2\Delta x}(-u_{j-1} + u_{j+1}) = 0$$

1. Assume a vector of 10 grid points of computed data
   $\overline{u} = [u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}]^T$
2. Boundary condition: $u_1 = u_{left}$

Difference Equations

$$\frac{\partial u_j}{\partial t} + \frac{a}{2\Delta x}\left(-u_{j-1} + u_{j+1}\right) = 0$$

$$\begin{array}{lccccccccccccc}
\frac{\partial u_2}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & u_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \} & -\frac{a}{2\Delta x}u_1 & = 0 \\
\frac{\partial u_3}{\partial t} + \frac{a}{2\Delta x}\{ & -u_2 & 0 & u_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \} & & = 0 \\
\frac{\partial u_4}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & -u_3 & 0 & u_5 & 0 & 0 & 0 & 0 & 0 & 0 & \} & & = 0 \\
\frac{\partial u_5}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & -u_4 & 0 & u_6 & 0 & 0 & 0 & 0 & 0 & \} & & = 0 \\
\frac{\partial u_6}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & 0 & -u_5 & 0 & u_7 & 0 & 0 & 0 & 0 & \} & & = 0 \\
\frac{\partial u_7}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & 0 & 0 & -u_6 & 0 & u_8 & 0 & 0 & 0 & \} & & = 0 \\
\frac{\partial u_8}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & 0 & 0 & 0 & -u_7 & 0 & u_9 & 0 & 0 & \} & & = 0 \\
\frac{\partial u_9}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & 0 & 0 & 0 & 0 & -u_8 & 0 & u_{10} & 0 & \} & & = 0 \\
\frac{\partial u_{10}}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u_9 & 0 & u_{11} & \} & & = 0 \\
\frac{\partial u_{11}}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u_{10} & 0 & \} & +\frac{a}{2\Delta x}u_{12} & = 0
\end{array}$$

1. BC: $u_1 = u_{left}$
2. Note the u_{12} does not exist, so we need to do something.

Difference Equations, With Correct BC

1. Replace equation for u_{11} with the 1^{st} backward difference scheme

$$\frac{\partial u_{11}}{\partial t} + \frac{a}{2\Delta x}\left[-2u_{10} + 2u_{11}\right] = 0$$

$$
\begin{array}{lcccccccccccccc}
\frac{\partial u_2}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & u_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \} & -\frac{a}{2\Delta x}u_{left} & = 0 \\
\frac{\partial u_3}{\partial t} + \frac{a}{2\Delta x}\{ & -u_2 & 0 & u_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \} & & = 0 \\
\frac{\partial u_4}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & -u_3 & 0 & u_5 & 0 & 0 & 0 & 0 & 0 & 0 & \} & & = 0 \\
\frac{\partial u_5}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & -u_4 & 0 & u_6 & 0 & 0 & 0 & 0 & 0 & \} & & = 0 \\
\frac{\partial u_6}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & 0 & -u_5 & 0 & u_7 & 0 & 0 & 0 & 0 & \} & & = 0 \\
\frac{\partial u_7}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & 0 & 0 & -u_6 & 0 & u_8 & 0 & 0 & 0 & \} & & = 0 \\
\frac{\partial u_8}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & 0 & 0 & 0 & -u_7 & 0 & u_9 & 0 & 0 & \} & & = 0 \\
\frac{\partial u_9}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & 0 & 0 & 0 & 0 & -u_8 & 0 & u_{10} & 0 & \} & & = 0 \\
\frac{\partial u_{10}}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u_9 & 0 & u_{11} & \} & & = 0 \\
\frac{\partial u_{11}}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2u_{10} & 2u_{11} & \} & & = 0
\end{array}
$$

Matrix Form, Second Order Central Differencing

The previous set of equations can be rewitten in a matrix form:

$$\frac{\partial \overline{u}}{\partial t} + \frac{a}{2\Delta x}\,[A]\,\overline{u} + \overline{bc} = \overline{0}$$

with

$$[A] = \begin{Bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 \end{Bmatrix}$$

and $\overline{bc} = \left[-\frac{a}{2\Delta x} u_{left}, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T$

A Coupled System of ODE's

Matrix Notation: $A = B(10 : \overline{b}, \overline{c}, 1)$ with
$\overline{b} = [-1, -1, -1, -1, -1, -1, -1, -1, -2]$ and $\overline{c} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 2]$

Matrix Form: Non-Periodic Diffusion Equation

Example 2: Diffusion Equation, non-periodic, 11 Grid Points

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$$

$$BC : u(0,t) = u_{left}, u(1,t) = u_{right}$$

$$IC : u(x,0) = g(x)$$

Second Order Central Differencing

$$\frac{\partial u_j}{\partial t} = \frac{\mu}{\Delta x^2}(u_{j-1} - 2u_j + u_{j+1})$$

Assume a vector of 9 grid point of computed data

$\overline{u} = [u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}]^T$

Boundary condition: $u_1 = u_{left}$ and $u_{11} = u_{right}$

Difference Equations

$$\frac{\partial u_j}{\partial t} = \frac{\mu}{\Delta x^2}(u_{j-1} - 2u_j + u_{j+1})$$

$$\begin{array}{rcl}
\frac{\partial u_2}{\partial t} &=& \frac{\mu}{\Delta x^2}\{ \quad -2u_2 \quad u_3 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \} \quad + \frac{\mu}{\Delta x^2} u_{left} \\
\frac{\partial u_3}{\partial t} &=& \frac{\mu}{\Delta x^2}\{ \quad u_2 \quad -2u_3 \quad u_4 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \} \\
\frac{\partial u_4}{\partial t} &=& \frac{\mu}{\Delta x^2}\{ \quad 0 \quad u_3 \quad -2u_4 \quad u_5 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \} \\
\frac{\partial u_5}{\partial t} &=& \frac{\mu}{\Delta x^2}\{ \quad 0 \quad 0 \quad u_4 \quad -2u_5 \quad u_6 \quad 0 \quad 0 \quad 0 \quad 0 \quad \} \\
\frac{\partial u_6}{\partial t} &=& \frac{\mu}{\Delta x^2}\{ \quad 0 \quad 0 \quad 0 \quad u_5 \quad -2u_6 \quad u_7 \quad 0 \quad 0 \quad 0 \quad \} \\
\frac{\partial u_7}{\partial t} &=& \frac{\mu}{\Delta x^2}\{ \quad 0 \quad 0 \quad 0 \quad 0 \quad u_6 \quad -2u_7 \quad u_8 \quad 0 \quad 0 \quad \} \\
\frac{\partial u_8}{\partial t} &=& \frac{\mu}{\Delta x^2}\{ \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad u_7 \quad -2u_8 \quad u_9 \quad 0 \quad \} \\
\frac{\partial u_9}{\partial t} &=& \frac{\mu}{\Delta x^2}\{ \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad u_8 \quad -2u_9 \quad u_{10} \quad \} \\
\frac{\partial u_{10}}{\partial t} &=& \frac{\mu}{\Delta x^2}\{ \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad u_9 \quad -2u_{10} \quad \} \quad + \frac{\mu}{\Delta x^2} u_{right}
\end{array}$$

Note boundary conditions are included.

Matrix Form, Second Order Central Differencing

The previous set of equations can be rewitten in a matrix form:

$$\frac{\partial \overline{u}}{\partial t} = \frac{\mu}{\Delta x^2} [A] \overline{u} + \overline{bc}$$

with

$$[A] = \begin{Bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{Bmatrix}$$

and $\overline{bc} = \frac{\mu}{\Delta x^2} [u_{left}, 0, 0, 0, 0, 0, 0, 0, u_{right}]^T$

A Coupled System of ODE's

Matrix Notation: $A = B(9 : 1, -2, 1)$

Matrix Form: Periodic Wave Equation

Example 3: Linear Wave Equation, periodic, 11 Grid Points

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$BC : u(0,t) = u(2\pi,t) \qquad IC : u(x,0) = g(x)$$

Second Order Central Differencing

$$\frac{\partial u_j}{\partial t} + \frac{a}{2\Delta x}(-u_{j-1} + u_{j+1}) = 0$$

Assume a vector of 11 grid points of computed data

$\overline{u} = [u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}]^T$

Difference Equations

$$\frac{\partial u_j}{\partial t} + \frac{a}{2\Delta x}\left(-u_{j-1} + u_{j+1}\right) = 0$$

$$\begin{array}{lcccccccccccc}
\frac{\partial u_1}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & u_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u_{11} & \} = 0 \\
\frac{\partial u_2}{\partial t} + \frac{a}{2\Delta x}\{ & -u_1 & 0 & u_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \} = 0 \\
\frac{\partial u_3}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & -u_2 & 0 & u_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \} = 0 \\
\frac{\partial u_4}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & -u_3 & 0 & u_5 & 0 & 0 & 0 & 0 & 0 & 0 & \} = 0 \\
\frac{\partial u_5}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & 0 & -u_4 & 0 & u_6 & 0 & 0 & 0 & 0 & 0 & \} = 0 \\
\frac{\partial u_6}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & 0 & 0 & -u_5 & 0 & u_7 & 0 & 0 & 0 & 0 & \} = 0 \\
\frac{\partial u_7}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & 0 & 0 & 0 & -u_6 & 0 & u_8 & 0 & 0 & 0 & \} = 0 \\
\frac{\partial u_8}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & 0 & 0 & 0 & 0 & -u_7 & 0 & u_9 & 0 & 0 & \} = 0 \\
\frac{\partial u_9}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u_8 & 0 & u_{10} & 0 & \} = 0 \\
\frac{\partial u_{10}}{\partial t} + \frac{a}{2\Delta x}\{ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u_9 & 0 & u_{11} & \} = 0 \\
\frac{\partial u_{11}}{\partial t} + \frac{a}{2\Delta x}\{ & u_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u_{10} & 0 & \} = 0
\end{array}$$

1. Because of the periodic BC: At $j = 1$ we need $u_{j-1} = u_0 = u_{11}$
2. Likewise at $j = 11$ we need $u_{j+1} = u_{12} = u_1$

Matrix Form, Second Order Central Differencing

The previous set of equations can be rewitten in a matrix form:

$$\frac{\partial \overline{u}}{\partial t} + \frac{a}{2\Delta x}\,[A]\,\overline{u} = \overline{0}$$

with

$$[A] = \begin{Bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{Bmatrix}$$

A Coupled System of ODE's

Matrix Notation: $A = B_p(11 : -1, 0, 1)$

Diffusion Equation: Neumann BC

Example 4: Diffusion Equation, Right BC Neunamm, 11 Grid Points

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$$

$$BC : u(0,t) = u_{left}, \frac{\partial u(1,t)}{\partial x} = \left(\frac{\partial u}{\partial x}\right)_b$$

$$IC : u(x,0) = g(x)$$

Second Order Central Differencing

$$\frac{\partial u_j}{\partial t} = \frac{\mu}{\Delta x^2}(u_{j-1} - 2u_j + u_{j+1})$$

Assume a vector of 9 grid point of computed data

$\overline{u} = [u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}]^T$

At Right Boundary: Derivative is given instead of point value.

Boundary Condition Implementation

1. The Interior Difference Equations are the same as before, except for the equation at $j = 10$
2. Design of Boundary Operator for Neumann BC
   (a) At General Point j folding in the Neumann Condition with local point values

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j - \left[\frac{1}{\Delta x^2}(au_{j-1} + bu_j) + \frac{c}{\Delta x}\left(\frac{\partial u}{\partial x}\right)_{j+1}\right] = ?$$

	u_j	$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2 \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3 \cdot \left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4 \cdot \left(\frac{\partial^4 u}{\partial x^4}\right)_j$
$\Delta x^2 \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)_j$			1		
$-au_{j-1}$	$-a$	$-a \cdot (-1) \cdot \frac{1}{1}$	$-a \cdot (-1)^2 \cdot \frac{1}{2}$	$-a \cdot (-1)^3 \cdot \frac{1}{6}$	$-a \cdot (-1)^4 \cdot \frac{1}{24}$
$-bu_j$	$-b$				
$-\Delta x \cdot c \cdot \left(\frac{\partial u}{\partial x}\right)_{j+1}$		$-c$	$-c \cdot (1) \cdot \frac{1}{1}$	$-c \cdot (1)^2 \cdot \frac{1}{2}$	$-c \cdot (1)^3 \cdot \frac{1}{6}$
	—				

(b) Solving for a, b, and c, we obtain the following first-order operator, at $j = 10$:

$$(\delta_{xx}u)_{10} = \frac{1}{3\Delta x^2}(2u_9 - 2u_{10}) + \frac{2}{3\Delta x}\left(\frac{\partial u}{\partial x}\right)_{11}$$

Matrix Form

The previous set of equations can be rewitten in a matrix form:

$$\frac{\partial \overline{u}}{\partial t} = \frac{\mu}{\Delta x^2}\,[A]\,\overline{u} + \overline{bc}$$

with

$$[A] = \begin{Bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & -\frac{2}{3} \end{Bmatrix}$$

$\overline{bc} = \frac{\mu}{\Delta x^2}\left[u_{left}, 0, 0, 0, 0, 0, 0, 0, \frac{2\Delta x}{3}\left(\frac{\partial u}{\partial x}\right)_b\right]^T$

Matrix Notation: $A = B(9 : \overline{a}, \overline{b}, 1)$

with $\overline{a} = \left[1, 1, 1, 1, 1, 1, 1, 1, \frac{2}{3}\right]$ and
$\overline{b} = \left[-2, -2, -2, -2, -2, -2, -2, -2, -\frac{2}{3}\right]$

Matrix Form: Compact Schemes

1. For Example: the 3 point 4^{th} Order scheme

$$\left(\frac{\partial u}{\partial x}\right)_{j-1} + 4\left(\frac{\partial u}{\partial x}\right)_{j} + \left(\frac{\partial u}{\partial x}\right)_{j+1} - \frac{3}{\Delta x}(-u_{j-1} + u_{j+1}) = O(\Delta x^4)$$

2. In Matrix Form:

$$B(1,4,1)\delta_x \vec{u} = \frac{3}{\Delta x} B(-1,0,1)\vec{u} + \left(\vec{bc}\right)$$

in which Dirichlet boundary conditions (put into $\left(\vec{bc}\right)$) have been imposed.

3. Therefore, to apply the compact scheme
   (a) We need to form the right hand side above (A resulting vector)
   (b) Then, do a matrix inversion to get the resulting finite difference term $\frac{\partial u}{\partial x}_j$
   (c) $\delta_x \vec{u} = B(1,4,1)^{-1} \left[\frac{3}{\Delta x} B(-1,0,1)\vec{u} + \left(\vec{bc}\right) \right]$