C SANTA BARBARA
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Parallel fast sweeping: algorithms and applications

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UCSB Mechanical Engineering

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General static Hamilton-Jacobi equations

$$H(\nabla u(\mathbf{x}), \mathbf{x}, u(\mathbf{x})) = 1 \quad \text{for } \mathbf{x} \in \Omega \subset \mathbb{R}^n,$$
$$u(\mathbf{x}) = g(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma \subset \Omega.$$

M Falcone, E Cristiani, and S Cacace. A Local Ordered Upwind Method for Ha

A. Sethian and Alexander Vladimirsky. Ordered Upwind Methods for Static Hamilton–
Equations: Theory and Algorithms. SIAM Journal on Numerical Analysis, 41(1):325–
anuary 2003.



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Convex Hamilton-Jacobi Equations. Journal of Scientific Computing, 31(1-2):237–271, March 2007.

Björn Engquist, Brittany D. Froese, and Yen-Hsi Richard Tsai. Fast sweeping methods for
hyperbolic system
255:316–338, Dec

J. Sethian. A fas
Acad. Sci., 93:15

Y. Tsai, L. Cheng, S. Osher, and H. Zhao. Fast sweeping algorithms for a class of hamilto
jacobi equations. SIAM J. Numer. Anal., 41:673–694, 2003.

Outline

- The fast sweeping method (FSM)
- Parallel computing models
- Parallel fast sweeping on uniform grids
- Hybrid massively parallel algorithm
- Fast sweeping on nonuniform grids
- Applications: computation of isochrons
- Questions

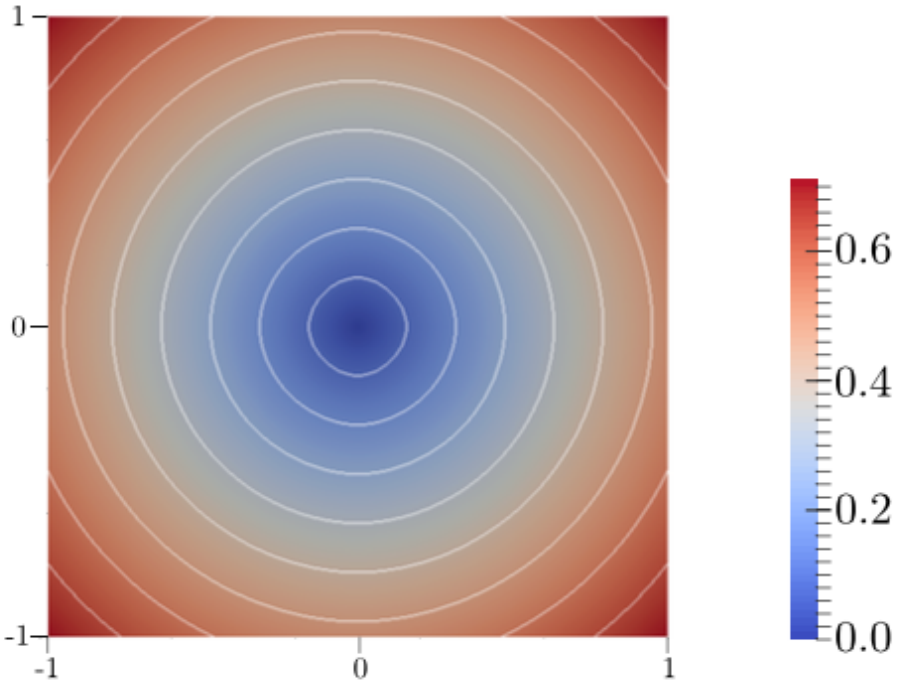
Outline

- **The fast sweeping method (FSM)**
  - Parallel computing models
  - Parallel fast sweeping on uniform grids
  - Hybrid massively parallel algorithm
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The Eikonal equation

- Nonlinear, hyperbolic equation
- Path planning, optimal control, image processing
- Solution is “arrival time” of wavefront
- Right hand side is inverse of speed

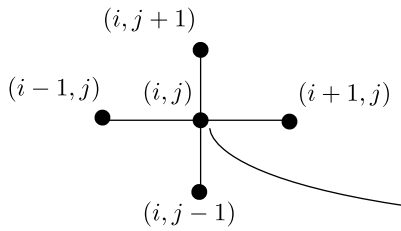
$$
\begin{aligned}
|\nabla u(\mathbf{x})| &= f(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega \subset \mathbb{R}^n, \\
u(\mathbf{x}) &= g(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma \subset \Omega,
\end{aligned}
$$

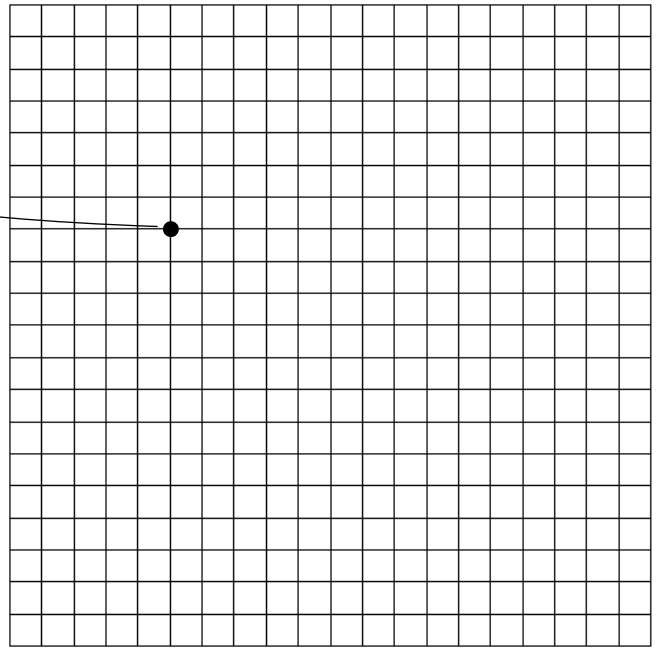


First computational method

- Finite difference method
  - Discrete equation satisfied at each point

$$[(u^h_{i,j} - u^h_{i-1,j})^+]^2 + [(u^h_{i,j} - u^h_{i+1,j})^+]^2$$
$$+ [(u^h_{i,j} - u^h_{i,j-1})^+]^2 + [(u^h_{i,j} - u^h_{i,j+1})^+]^2 = f^2_{i,j}h$$

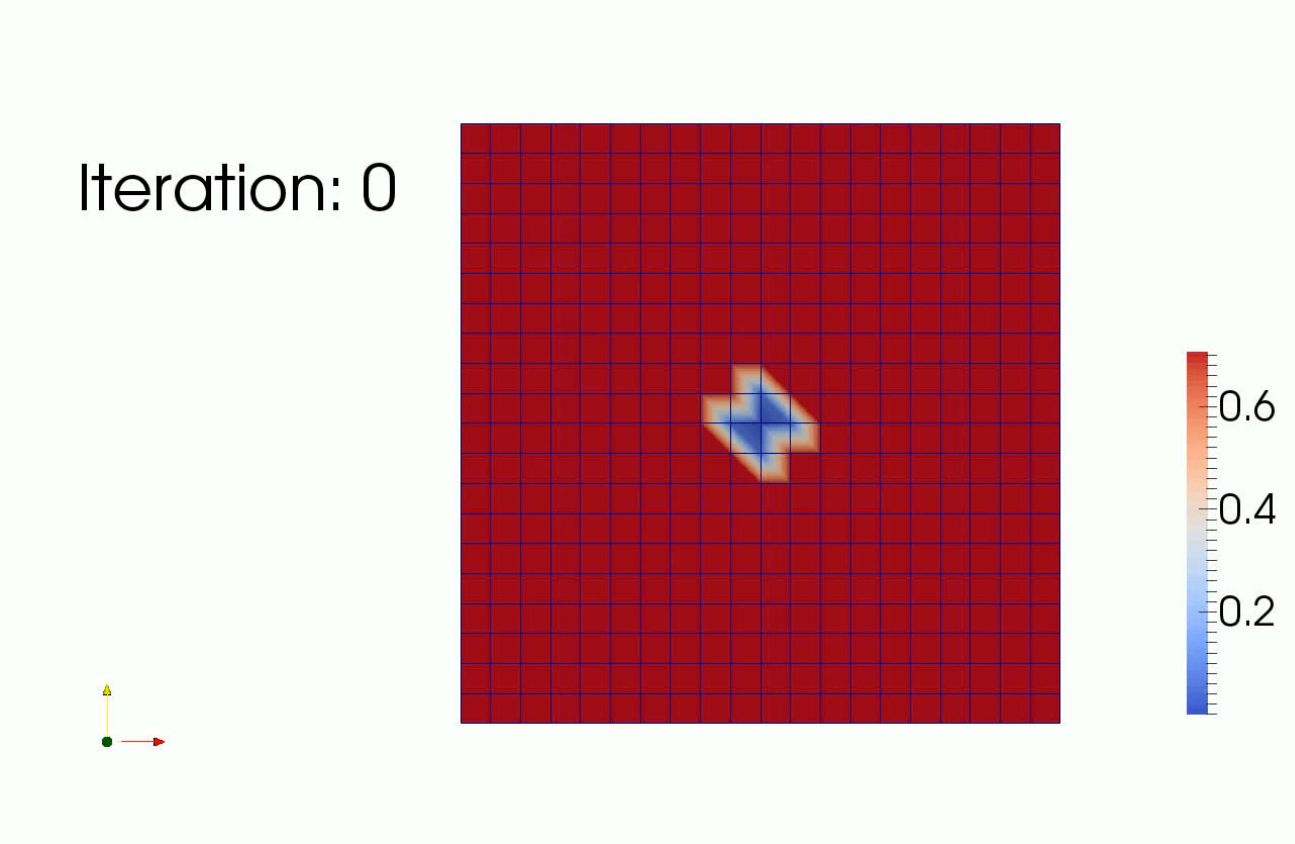




E. Rouy and A. Tourin, *A viscosity solution approach to shape-from-shading, SIAM J. Num. Anal.* **29** (June, 1992) 867–884.

erative method

- Update N points per iteration
- Maximum N iterations
  - $O(N^2)$ time complexity



A one minute problem

- 1992 Rouy-Tourin method
- 2D path planning problem
  - 200 x 200 grid
  - N=40,000

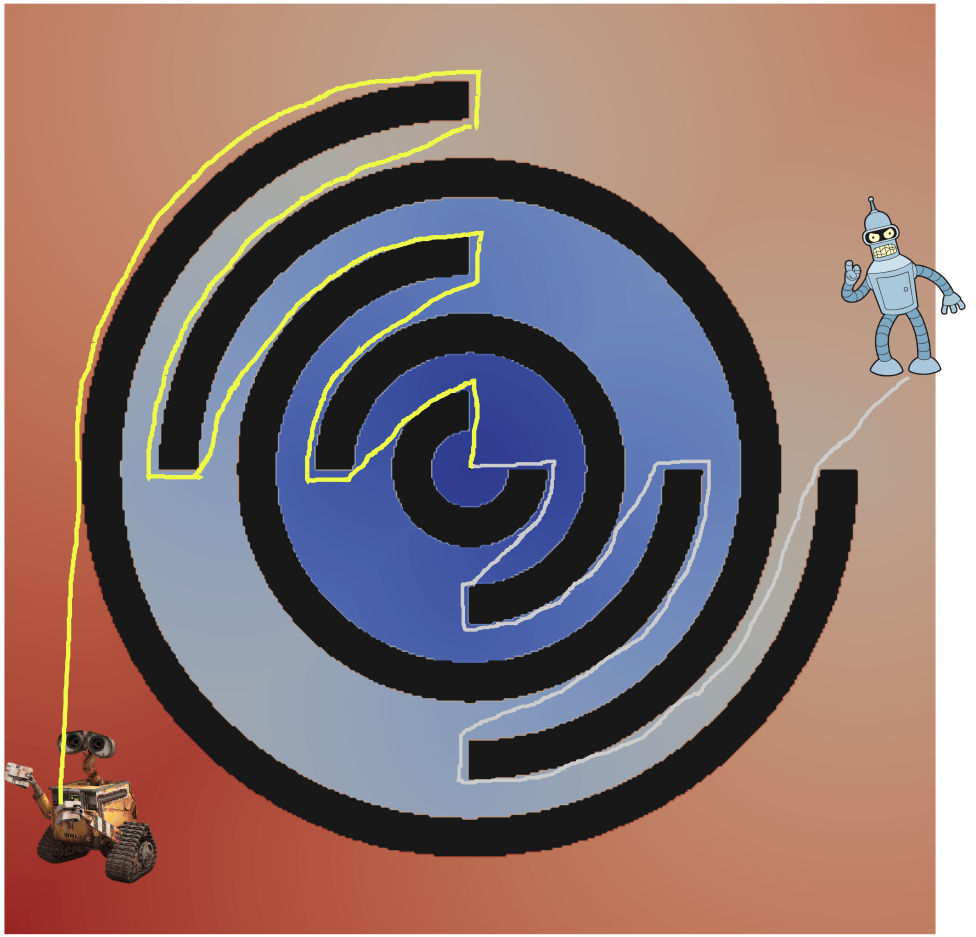
A one minute problem

- 1992 Rouy-Tourin method
- 2D path planning problem
  - 200 x 200 grid
  - N=40,000
- Solve Eikonal
  - Immediately have ALL optimal paths to the boundary set
  - Trace gradients backwards



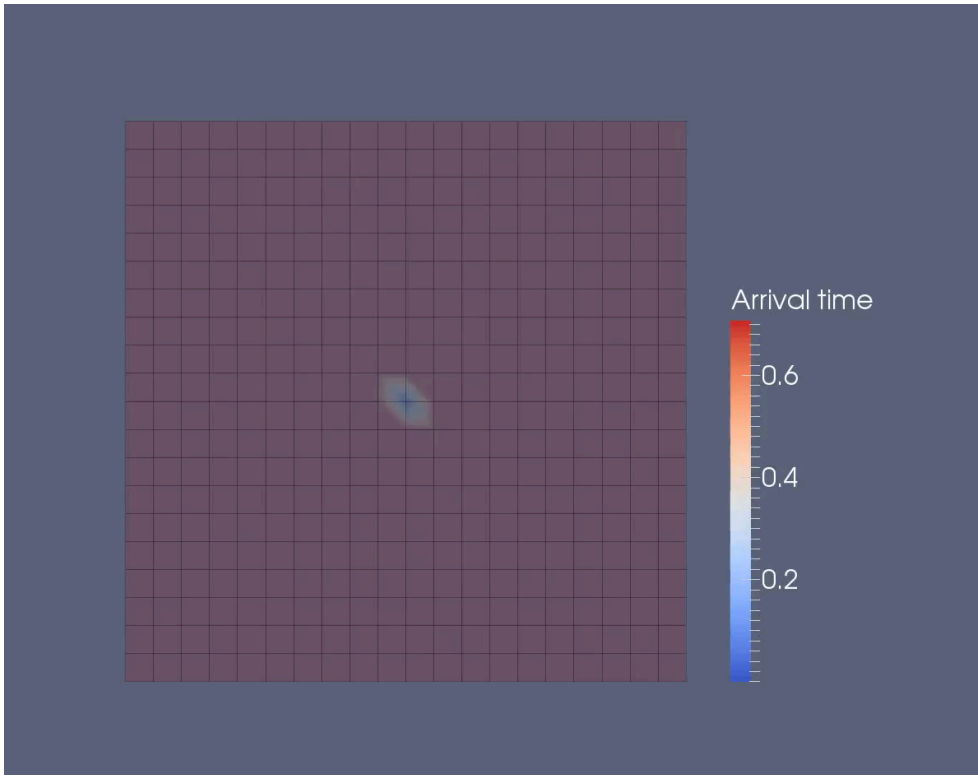
A one minute problem

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Fast marching method

- Same discrete equation
- Update points in order
  - "Single pass"
- O(Nlog(N)) worst case
- Still useful and popular
- Difficult to parallelize

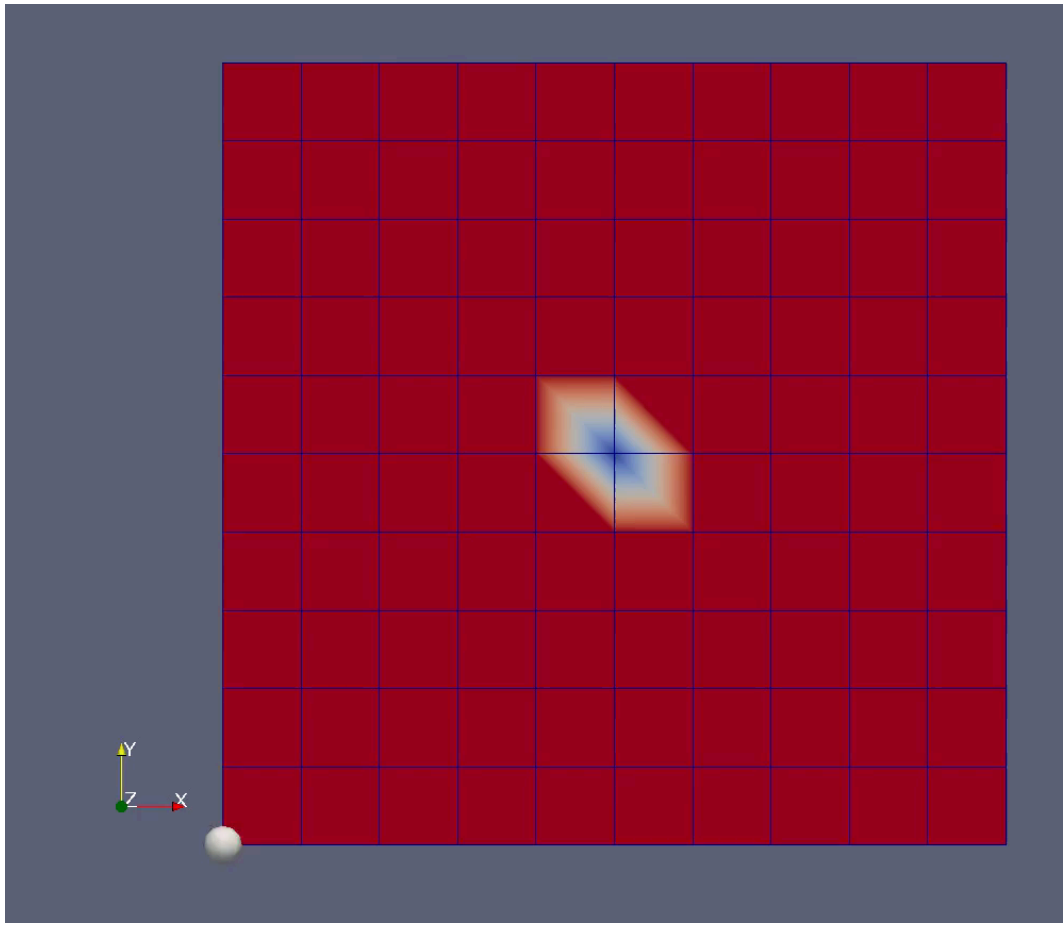


J. N. Tsitsiklis, *Efficient algorithms for globally optimal trajectories, IEEE Transactions on Automatic Control* **40** (1995) 1528–1538.

J. A. Sethian, *A Fast Marching Level Set Method for Monotonically Advancing Fronts., Proceedings of the National Academy of Sciences of the United States of America* **93** (Mar., 1996) 1591–5.

The Fast Sweeping Method

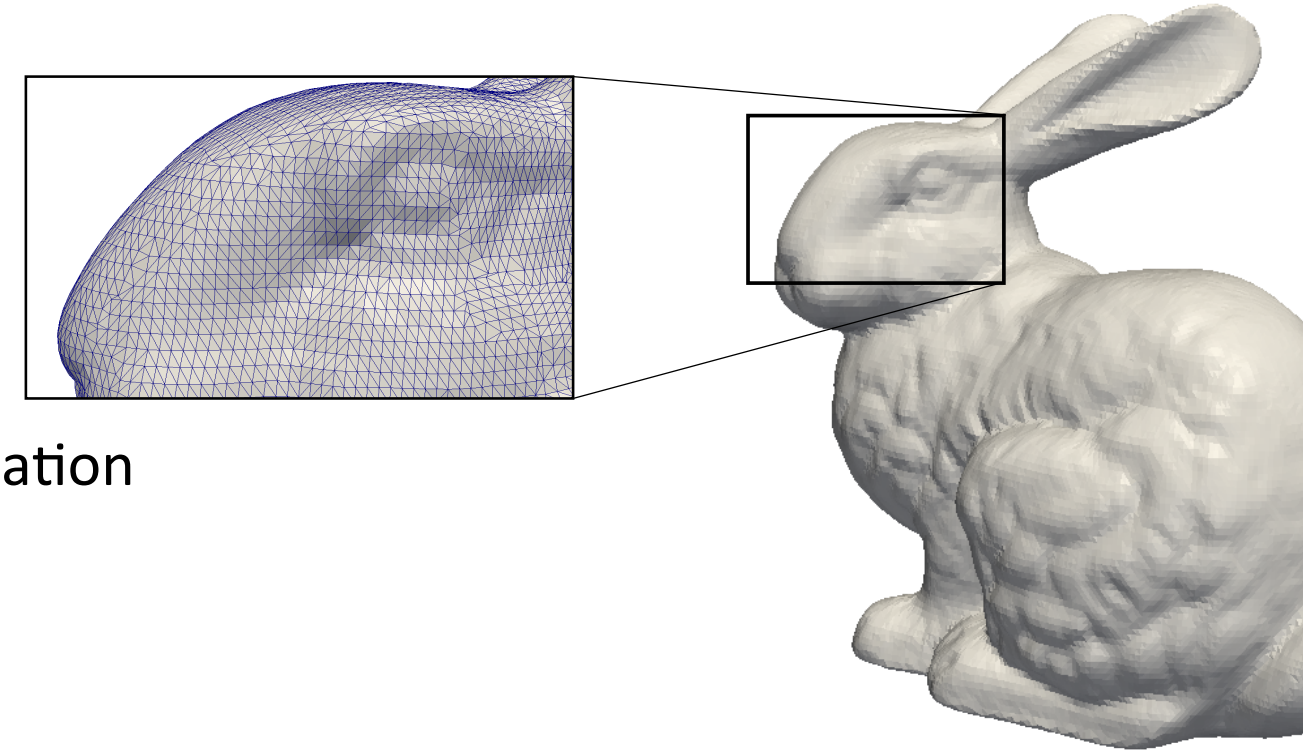
- Information propagates along characteristic curves
- A smarter iterative method
- Update grid points in a specific "sweeping order"
- O(N) time complexity



Zhao, H. A Fast Sweeping Method for Eikonal Equations. Mathematics of Computation 74, 250 (2004), 603–627.

One minute problem

- Fast sweeping method
- O(N) computations

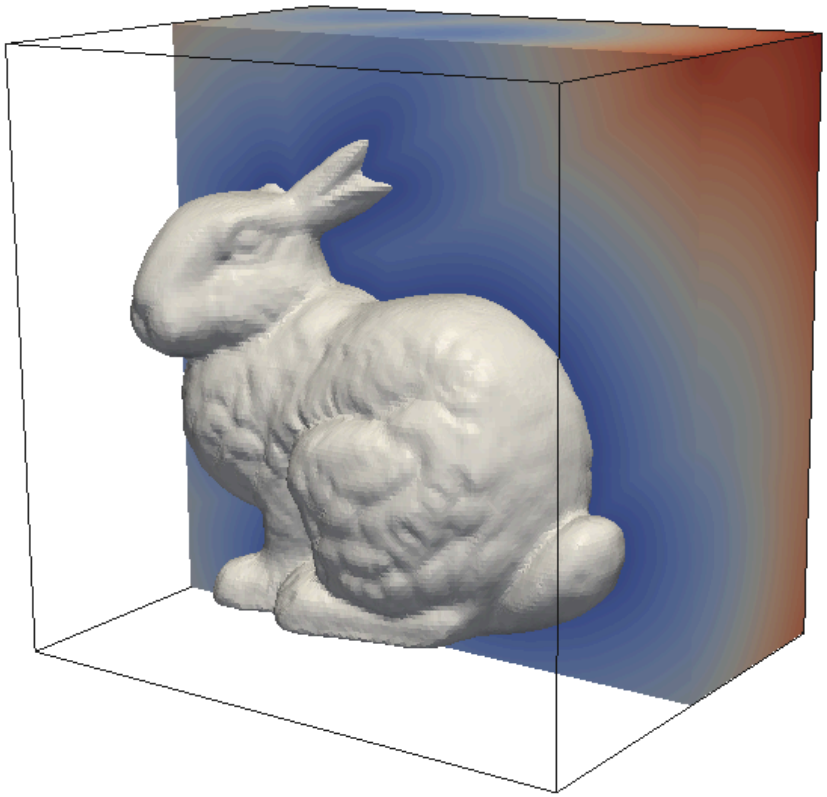


- Distance to a surface
- Gives an implicit representation
- Level set function

One minute problem

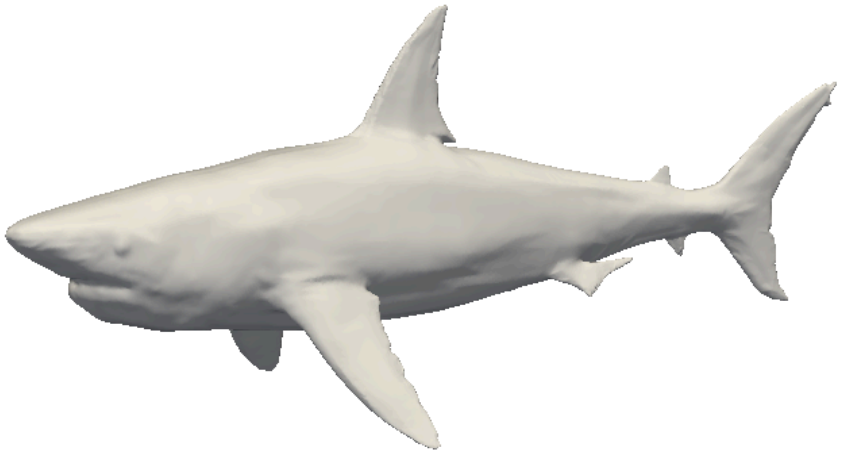
- Fast sweeping method
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➢One minute problem







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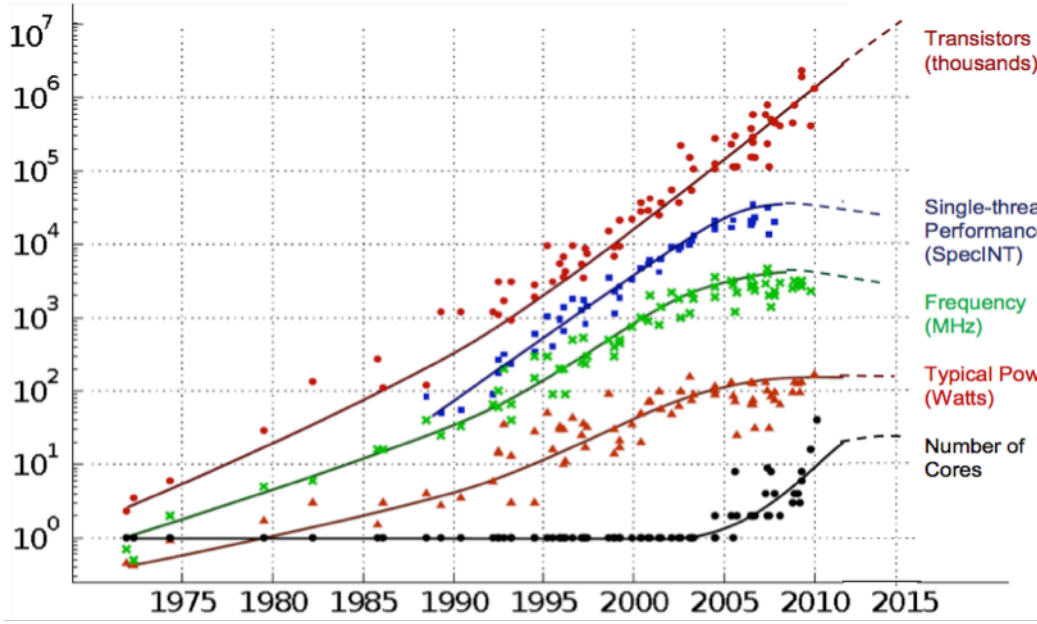


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Why parallel computing?

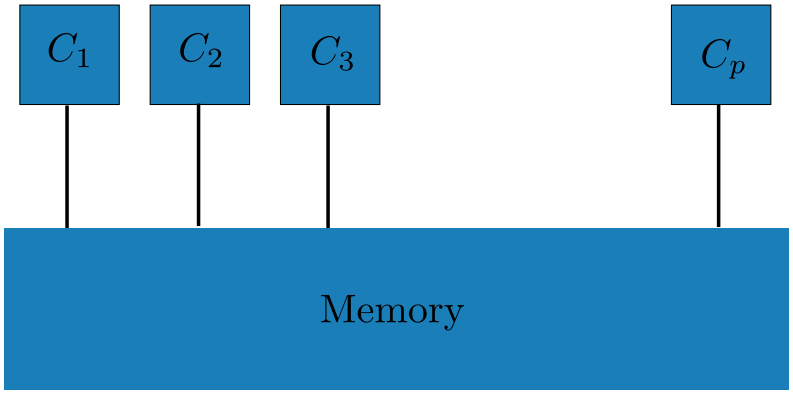
- Single core chips are no longer getting faster
- Multi-core is the new paradigm
  - Have to design new algorithms



Borrowed from Chuck Moore, AMD Corporate Fellow and CTO of Technology Group, presented at the 2011 Salishan Conference on High-speed Computing

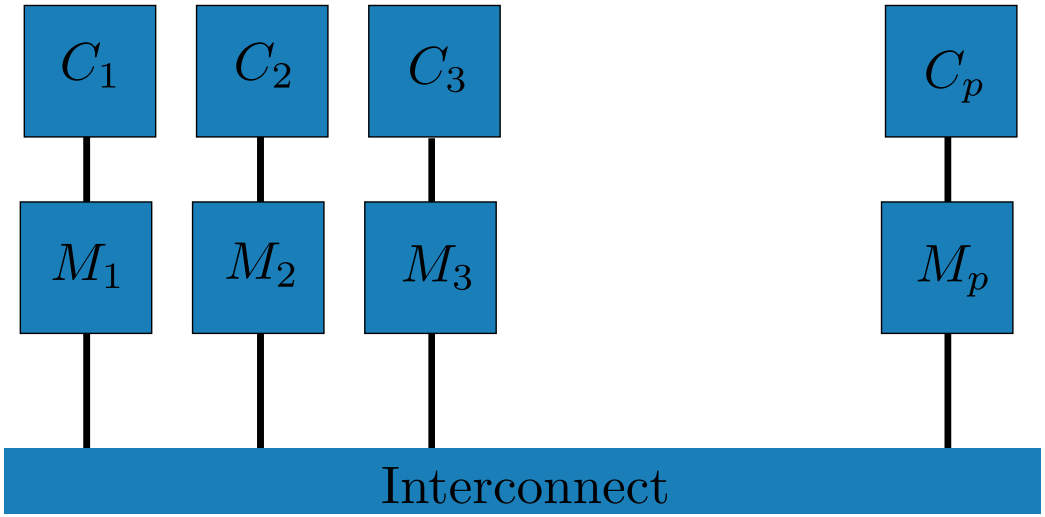
Basic parallel models

- Shared memory
  - Pros
    - Easy to program
    - Low parallel overhead
  - Cons
    - Relatively low processor counts
    - Relatively low memory



Basic parallel models

- Distributed memory
  - Pros
    - Can have a huge number of processors
    - Extreme amounts of memory
  - Cons
    - Communication through interconnect expensive
    - More complex to program

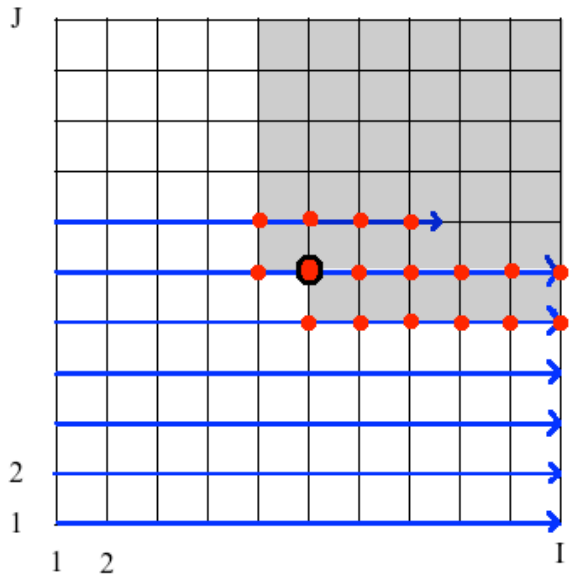


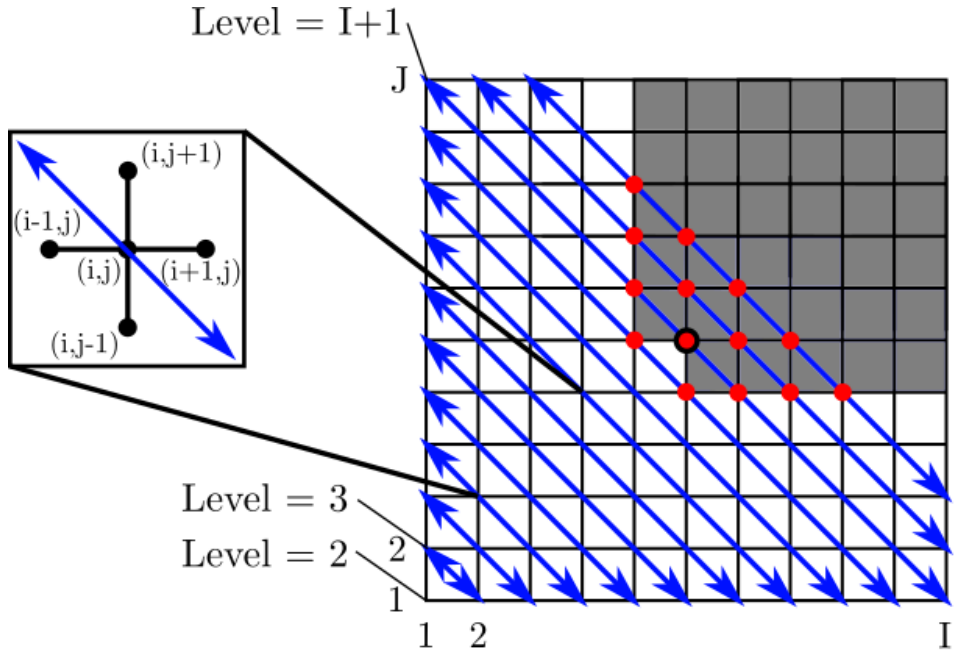
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Parallel Fast sweeping method

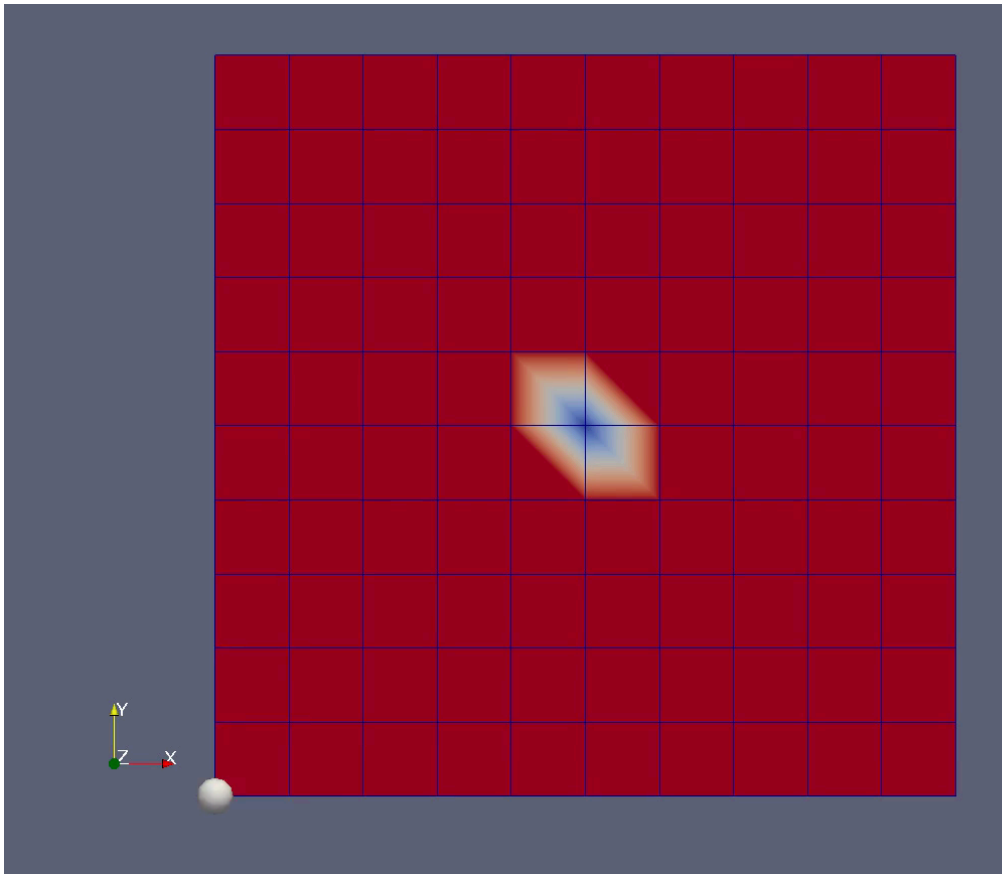
- FSM requires specific order
- There exist independent "levels"
- Suitable for shared memory only



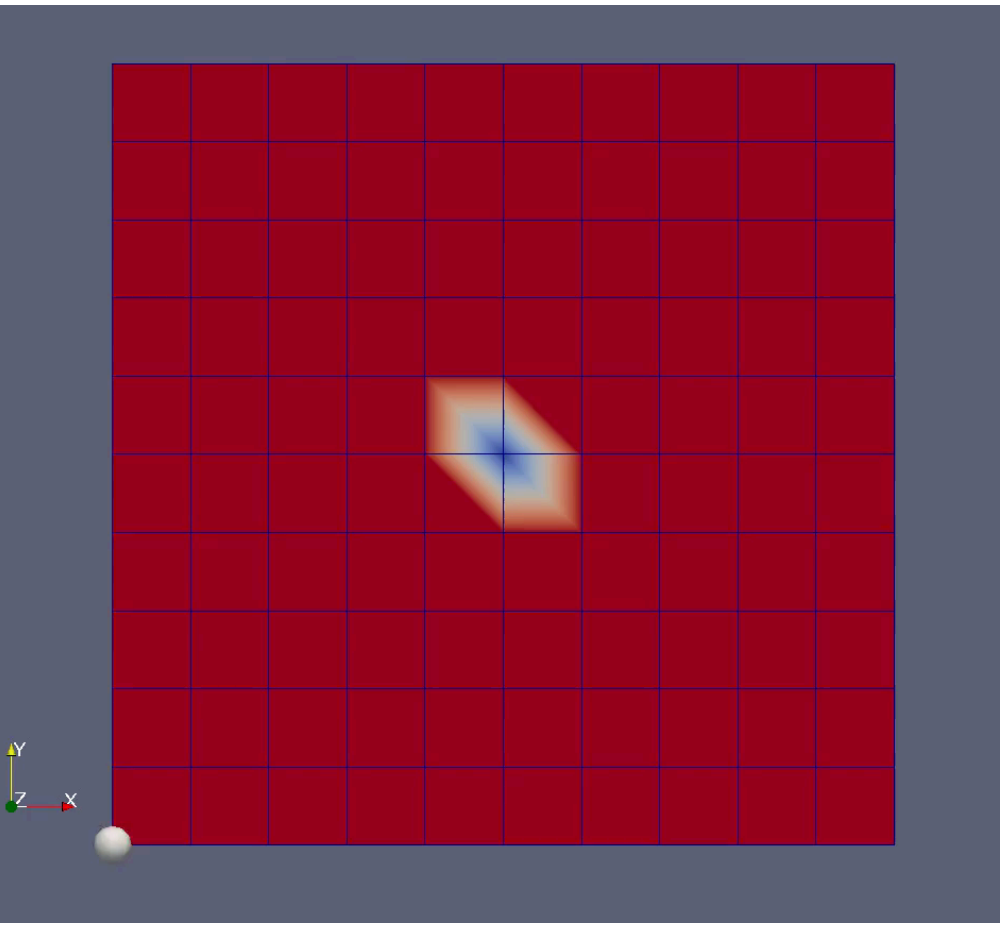


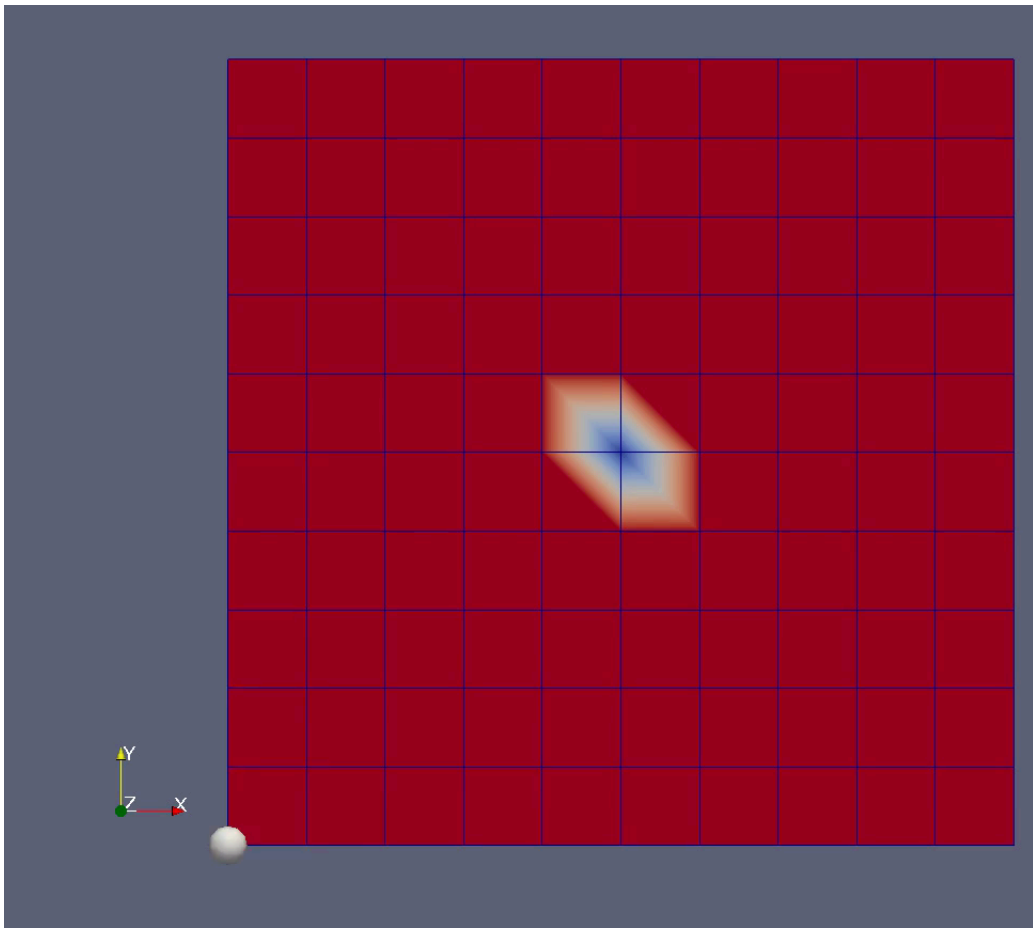
M. Detrixhe, F. Gibou, and C. Min, *A Parallel Fast Sweeping Method for the Eikonal Equation*, *Journal of Computational Physics* **237** (Mar., 2013) 46–55.

Parallel Fast sweeping method



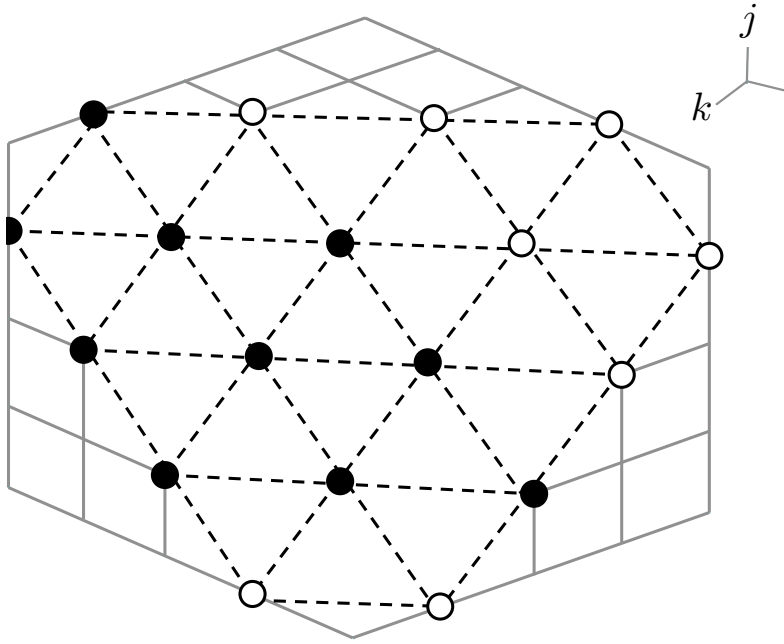
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Y
Z
X


Y
Z
X

Higher dimensions

- 3D parallel plane
- 4D hyperplane
- Parallelism increases with dimension

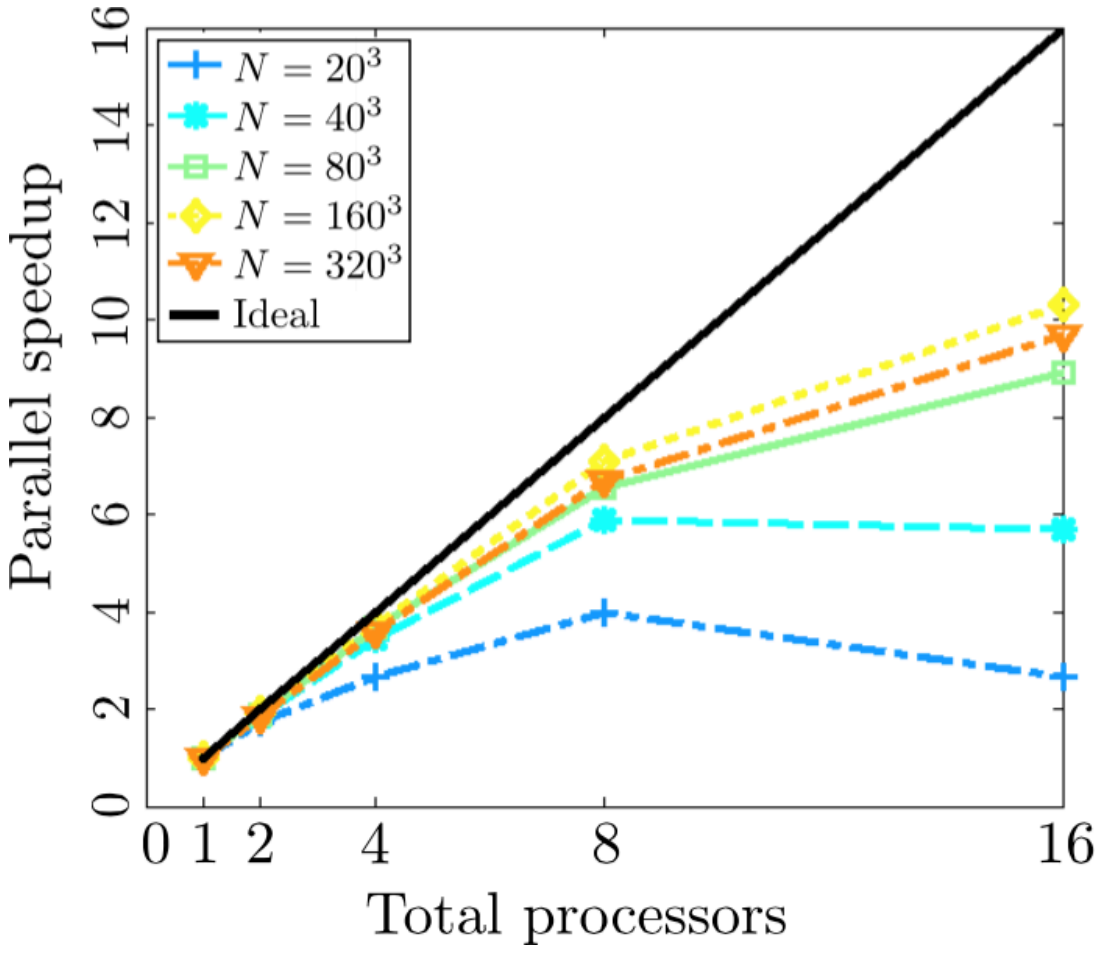


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Parallel speedup results

- Eikonal equation

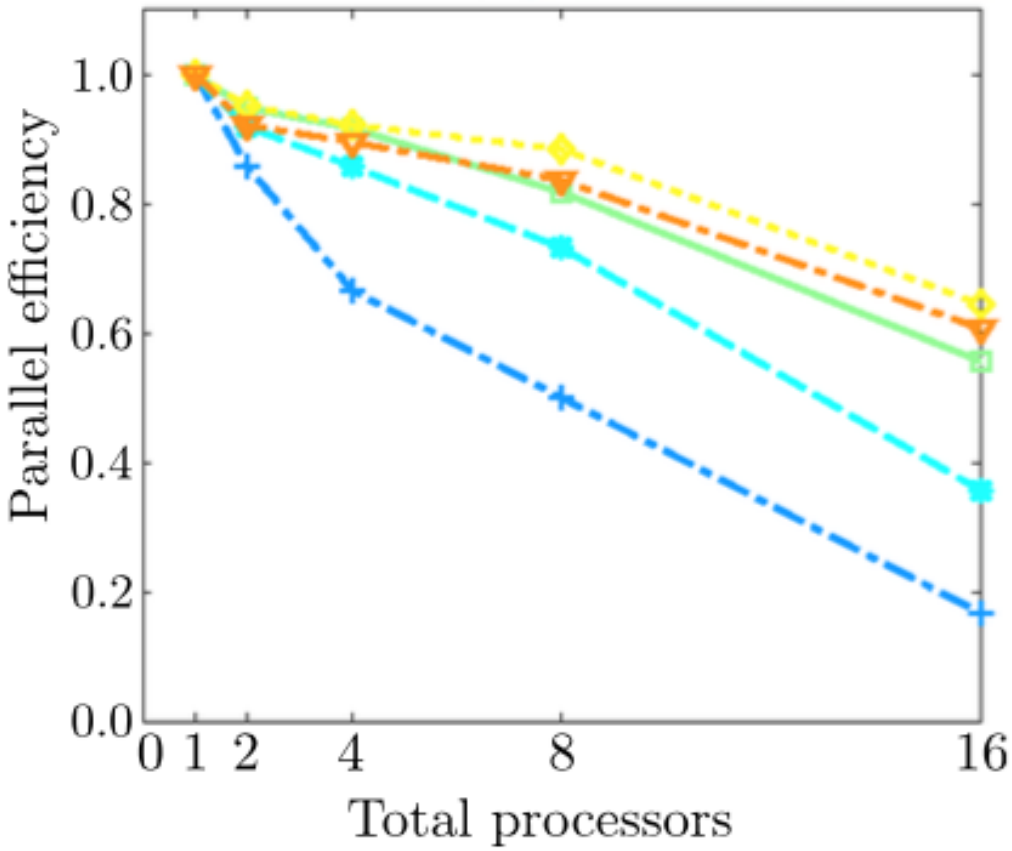
$$S_p = \frac{T_1}{T_p}$$



Parallel efficiency results

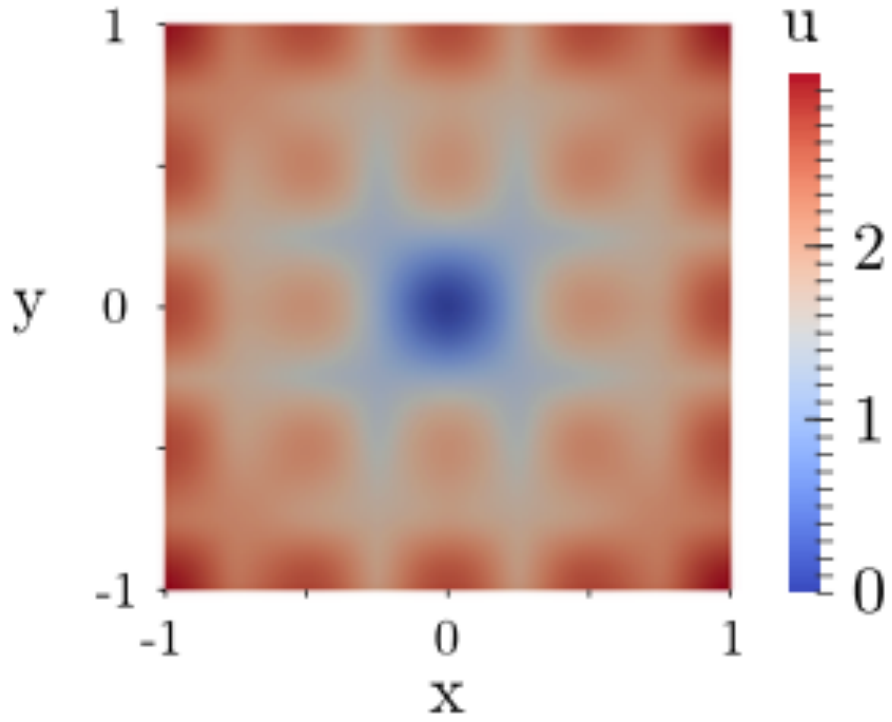
- Eikonal equation

$$E_p = \frac{T_1}{pT_p}$$

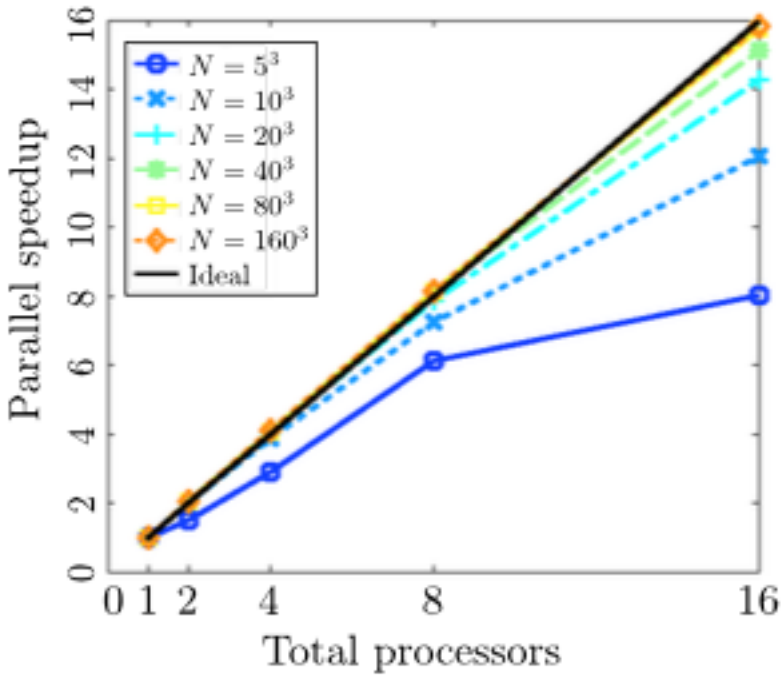


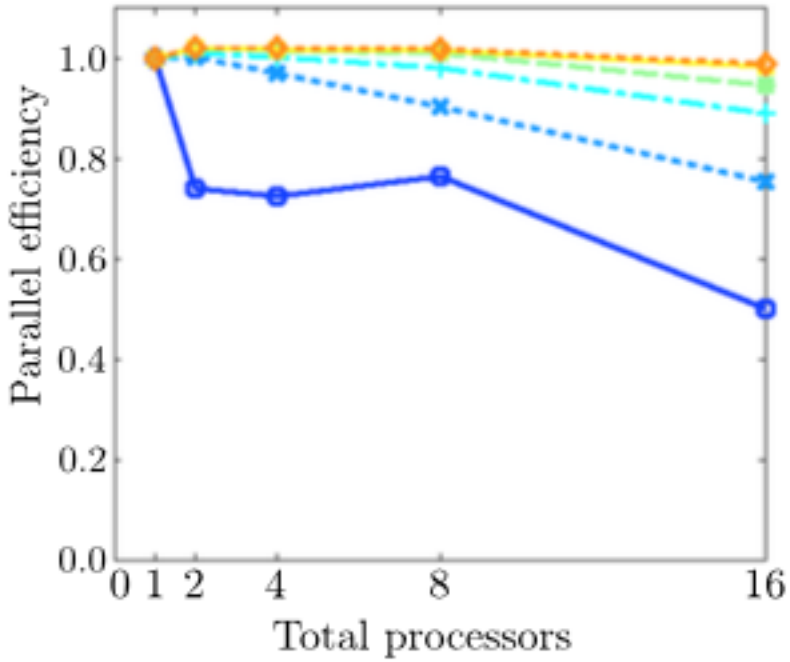
- Hamilton-Jacobi Bellman (HJB)

$$H(\mathbf{x}, \nabla u) = \min_{\mathbf{a} \in A} \{\nabla u(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}, \mathbf{a})\} = l(\mathbf{x}),$$
$$u(\mathbf{x}) = g(\mathbf{x}),$$



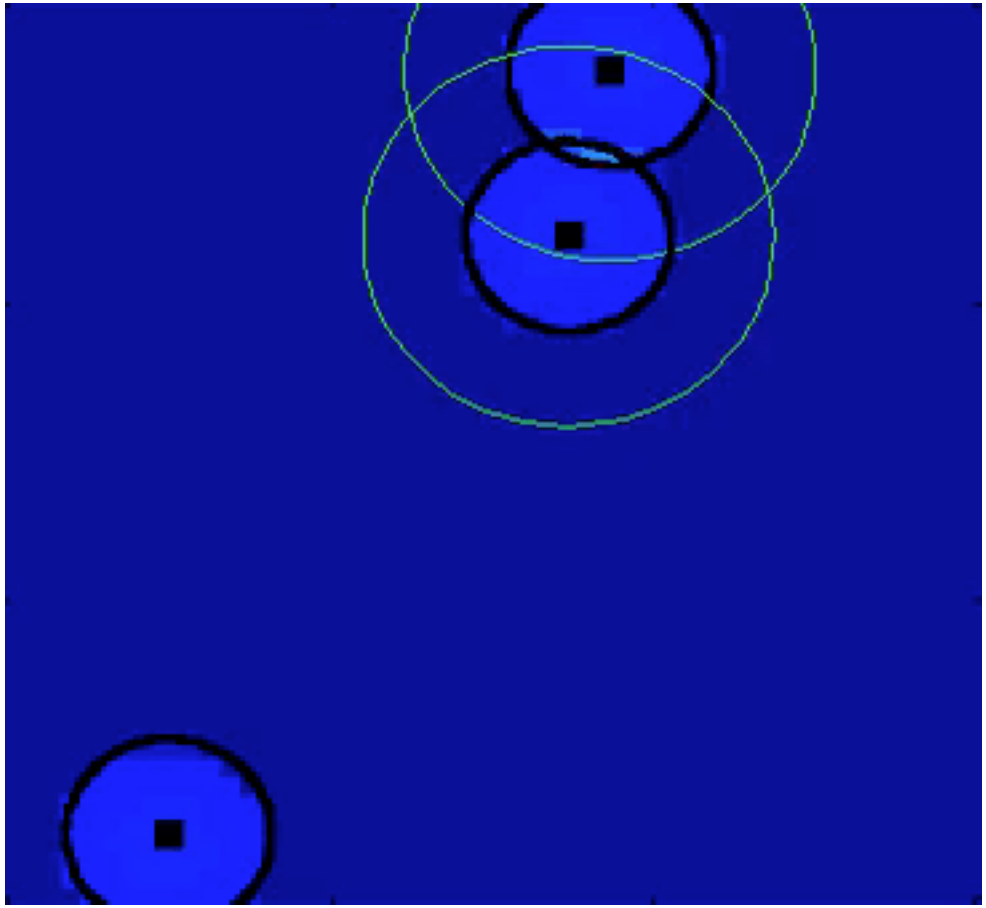
HJB Parallel results





One minute problem:

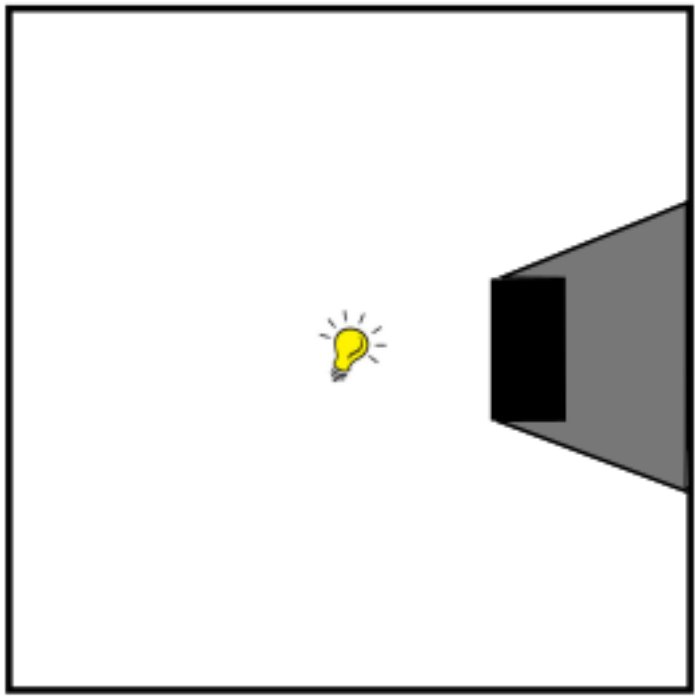
- Coverage control
- ”Cover” the entire domain, without collision
  - Roomba
  - Surveillance
  - Search and rescue

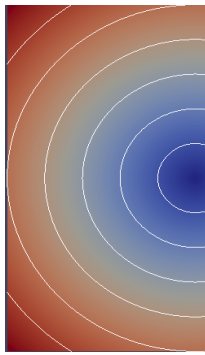


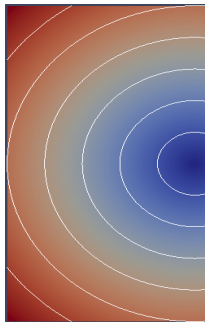
Hussein, I. I., & Stipanovic, D. M. (2007). Effective Coverage Control for Mobile Sensor Networks With Guaranteed Collision Avoidance. IEEE Transactions on Control Systems Technology, 15(4), 642–657. doi:10.1109/TCST.2007.899155

One minute problem

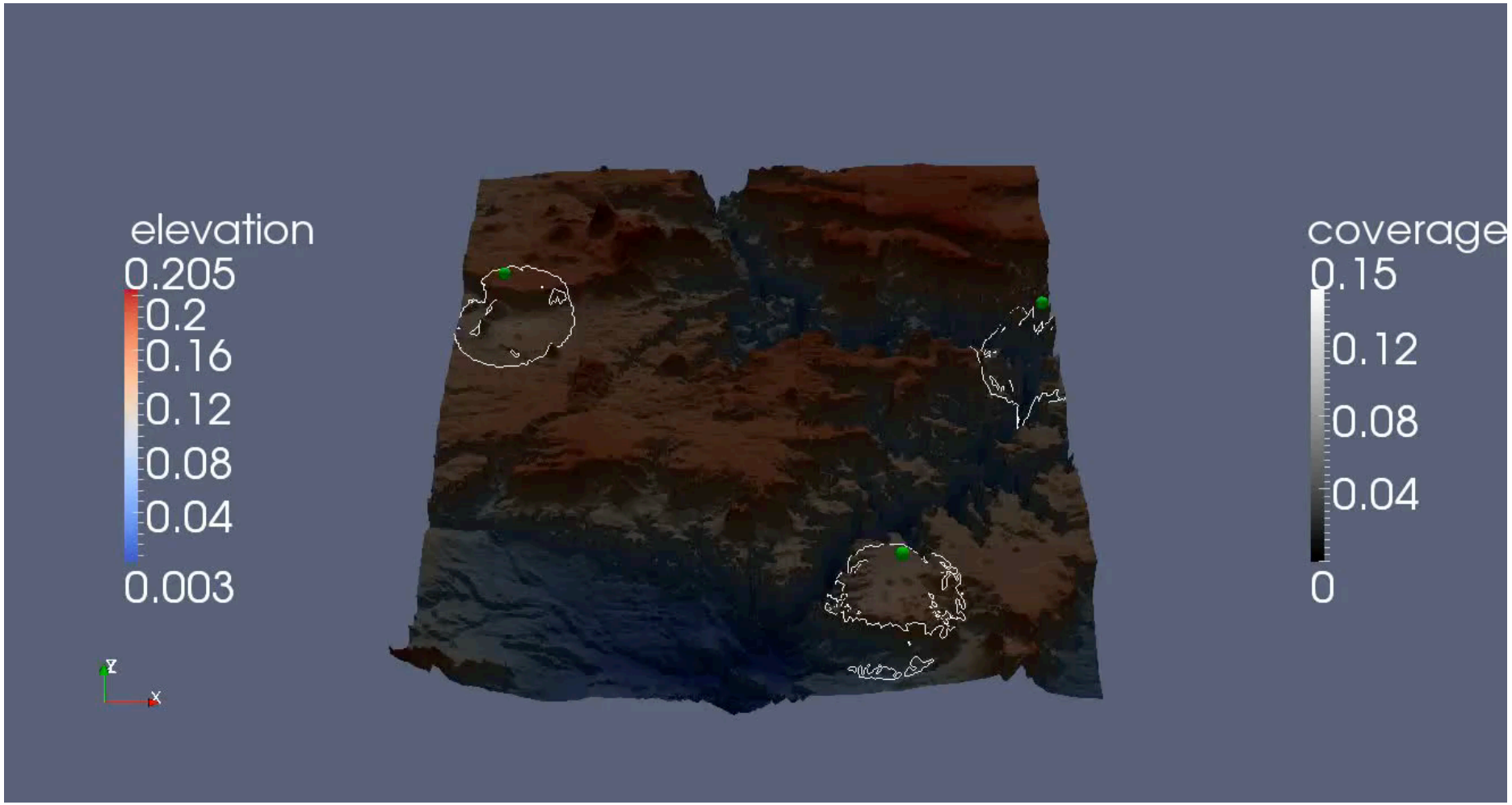
- Visibility region
  - Point source of light
  - Camera
  - Can determine from Eikonal









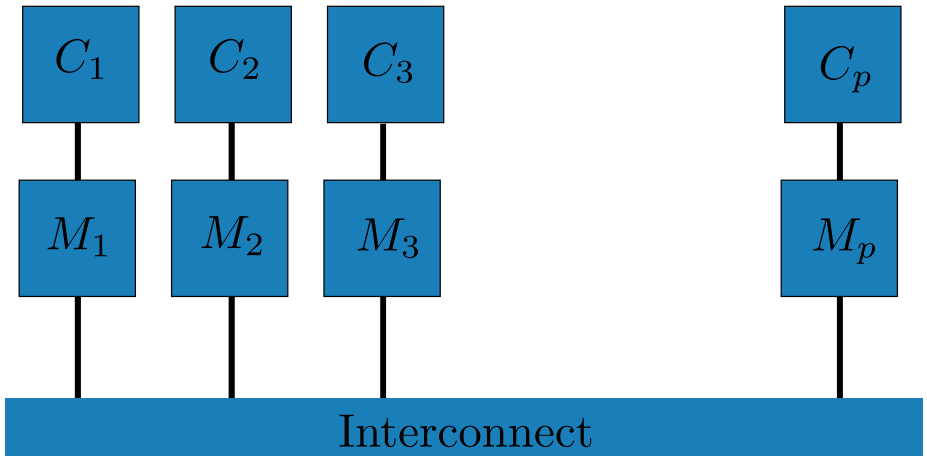

elevation
0.205
0.2
0.16
0.12
0.08
0.04
0.003
coverage
0.15
0.12
0.08
0.04
0
Y
X

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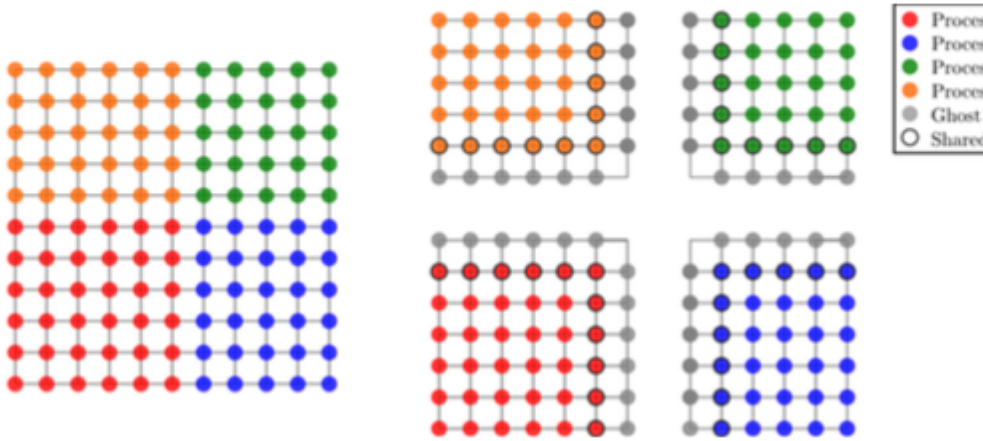
Shared memory is limiting

- Need to use a distributed memory method
  - Partition the domain and distribute among processors
  - Communication is expensive
    - 1,000-1,000,000 times more than compute



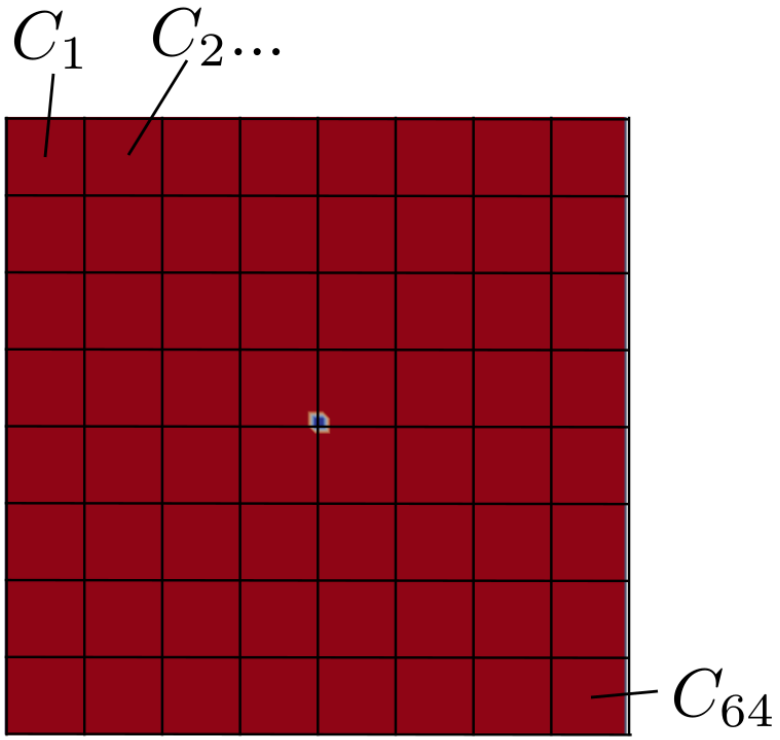
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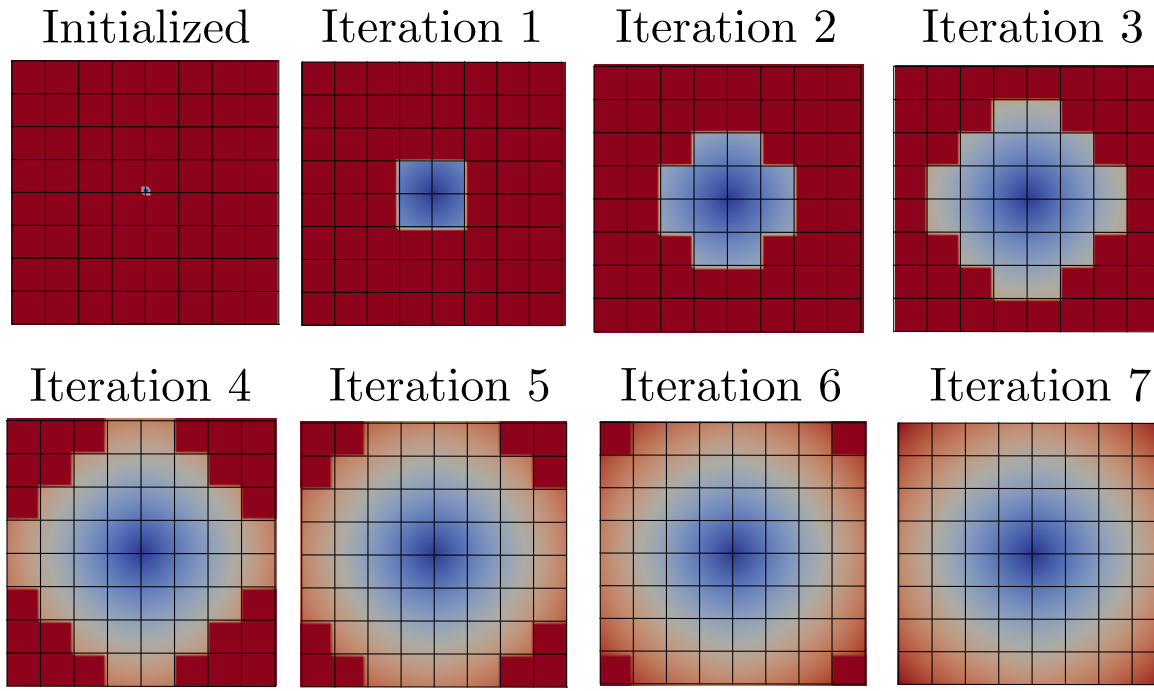
Domain decomposition

- Increases iterations
- Creates more work
- Can NOT achieve ideal speedup



Domain decomposition

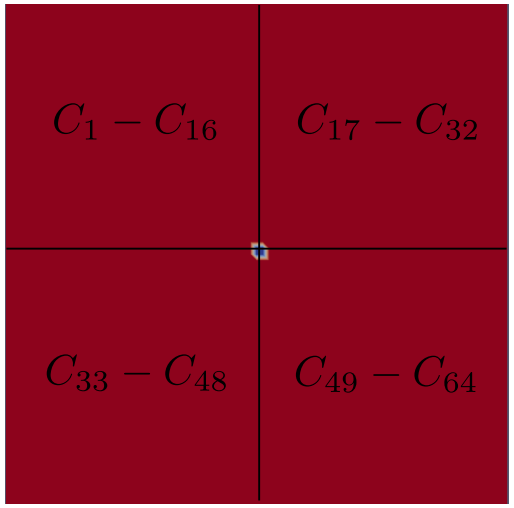
- Increases iterations
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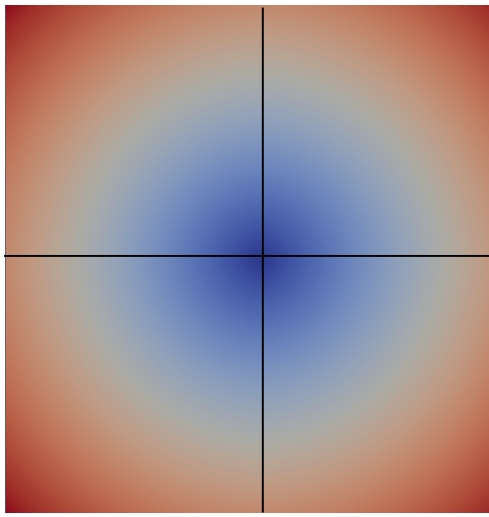
Hybrid method

- Same number of processors
- Much fewer iterations
- Completely scalable

Initialized

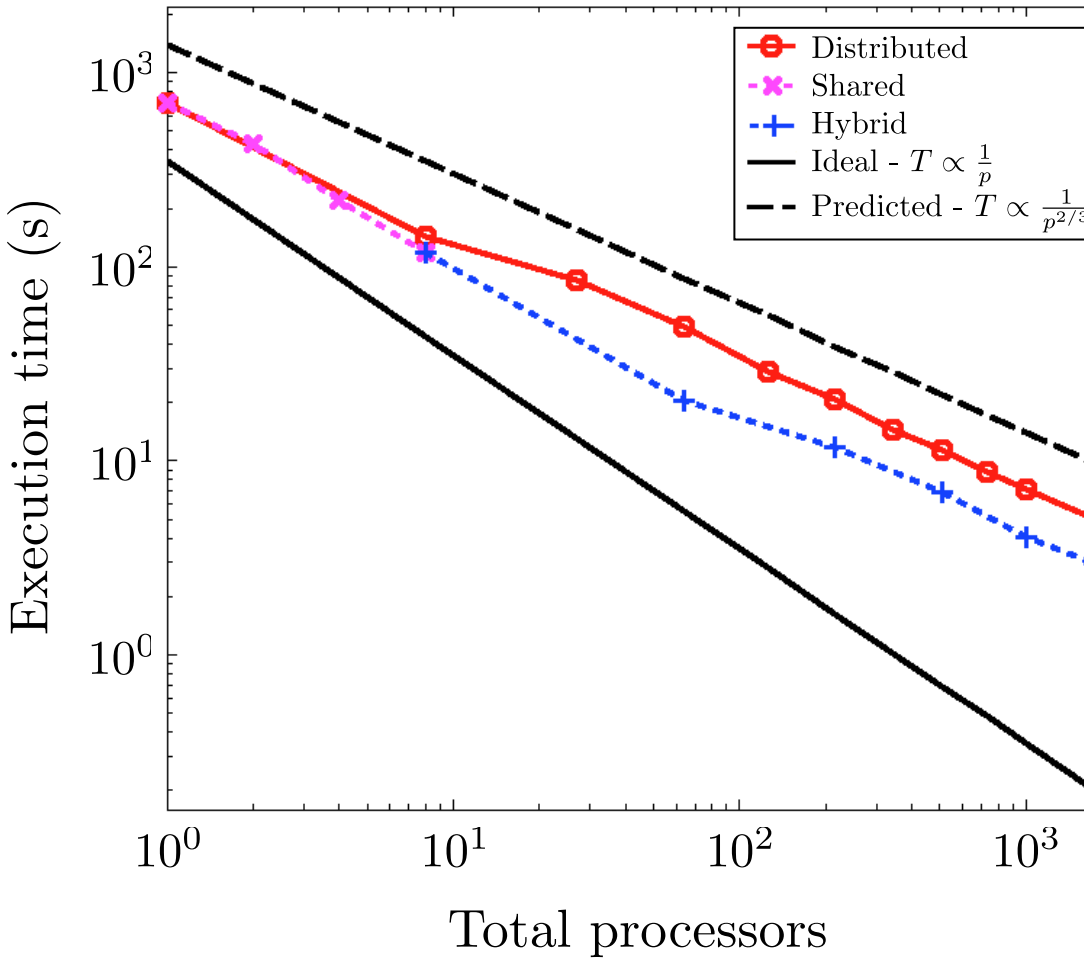


Iteration 1



Parallel results

- Both methods scale
- Hybrid method is better by a constant factor

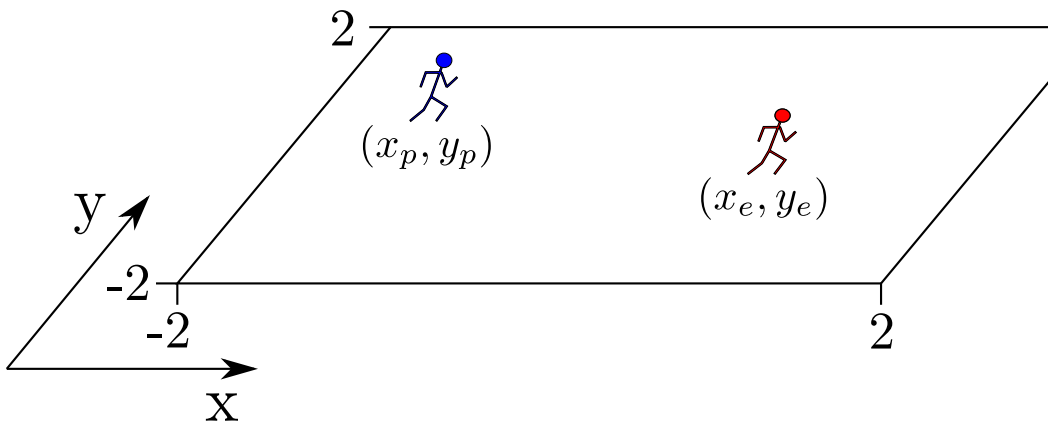


Detrixhe and F. Gibou, *Hybrid Massively Parallel Fast Sweeping Method for tic Hamilton-Jacobi Equations, In preparation* (2014).

One-minute problem

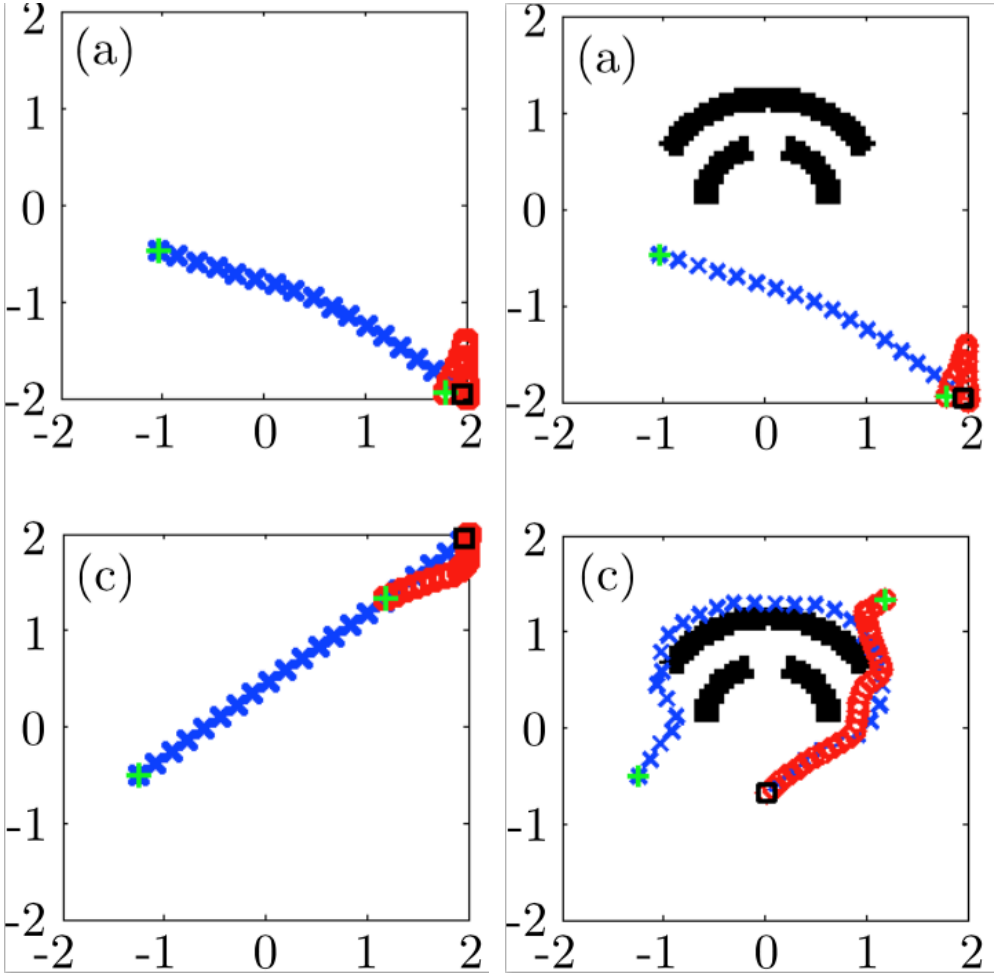
- Tag-chase game
- 4D problem
- Hamilton-Jacobi Issacs
- Expensive minimization

$$\max_{\mathbf{a}\in A}\min_{\mathbf{b}\in B}\{(\nabla u(\mathbf{x}))\cdot\mathbf{f}(\mathbf{x},\mathbf{a},\mathbf{b})\} = l(\mathbf{x})$$
$$u(\mathbf{x}) = g(\mathbf{x})$$



One minute problem

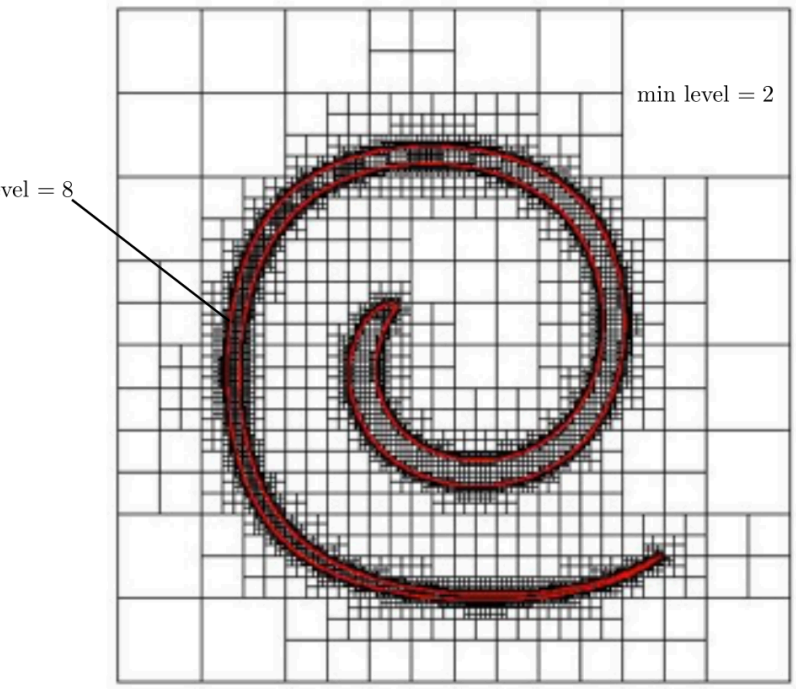
- 40x40x40x40 grid
  - 2.5M grid points
- 4096 processors
- Once solved, have ALL optimal trajectories for both players

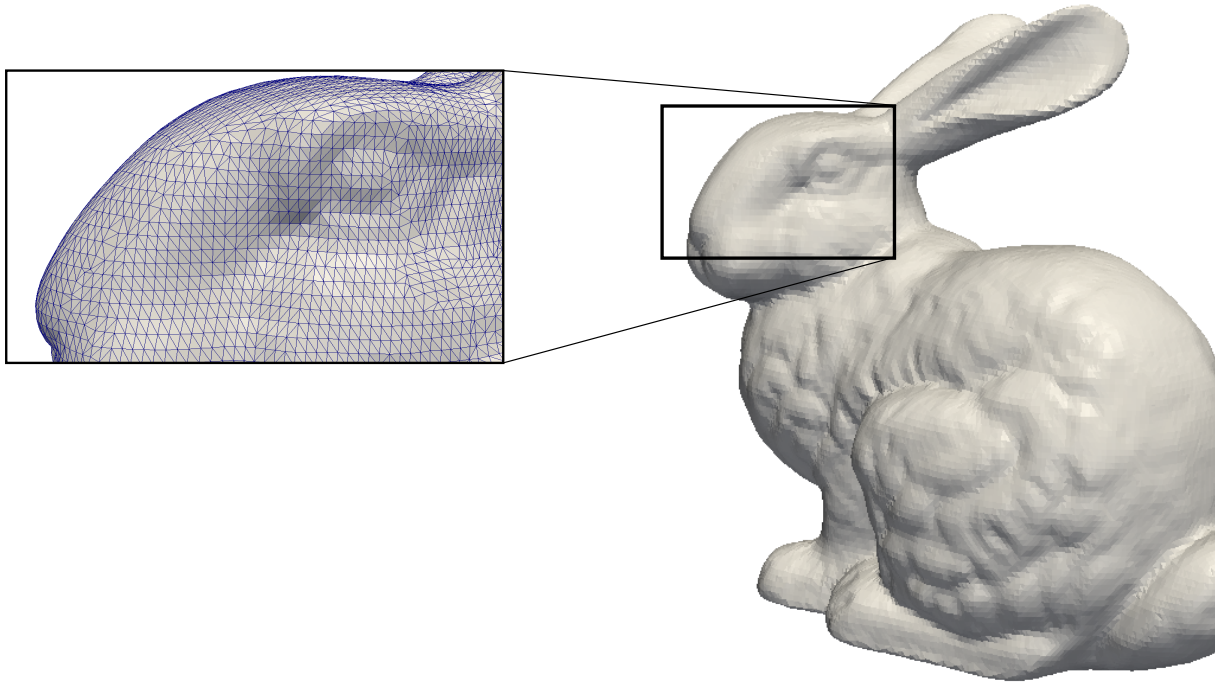


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Nonuniform grids





Fast sweeping on nonuniform grids

- Discrete equation at each point
- Specific sweeping order

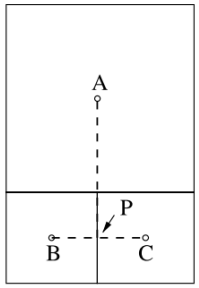
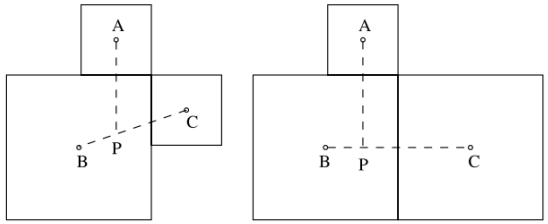
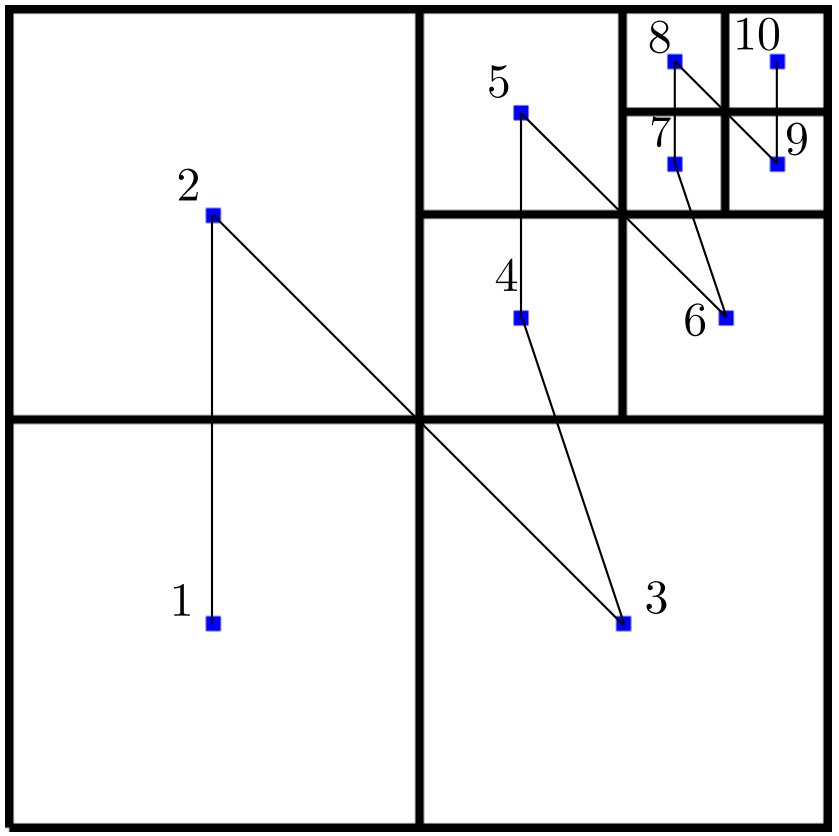
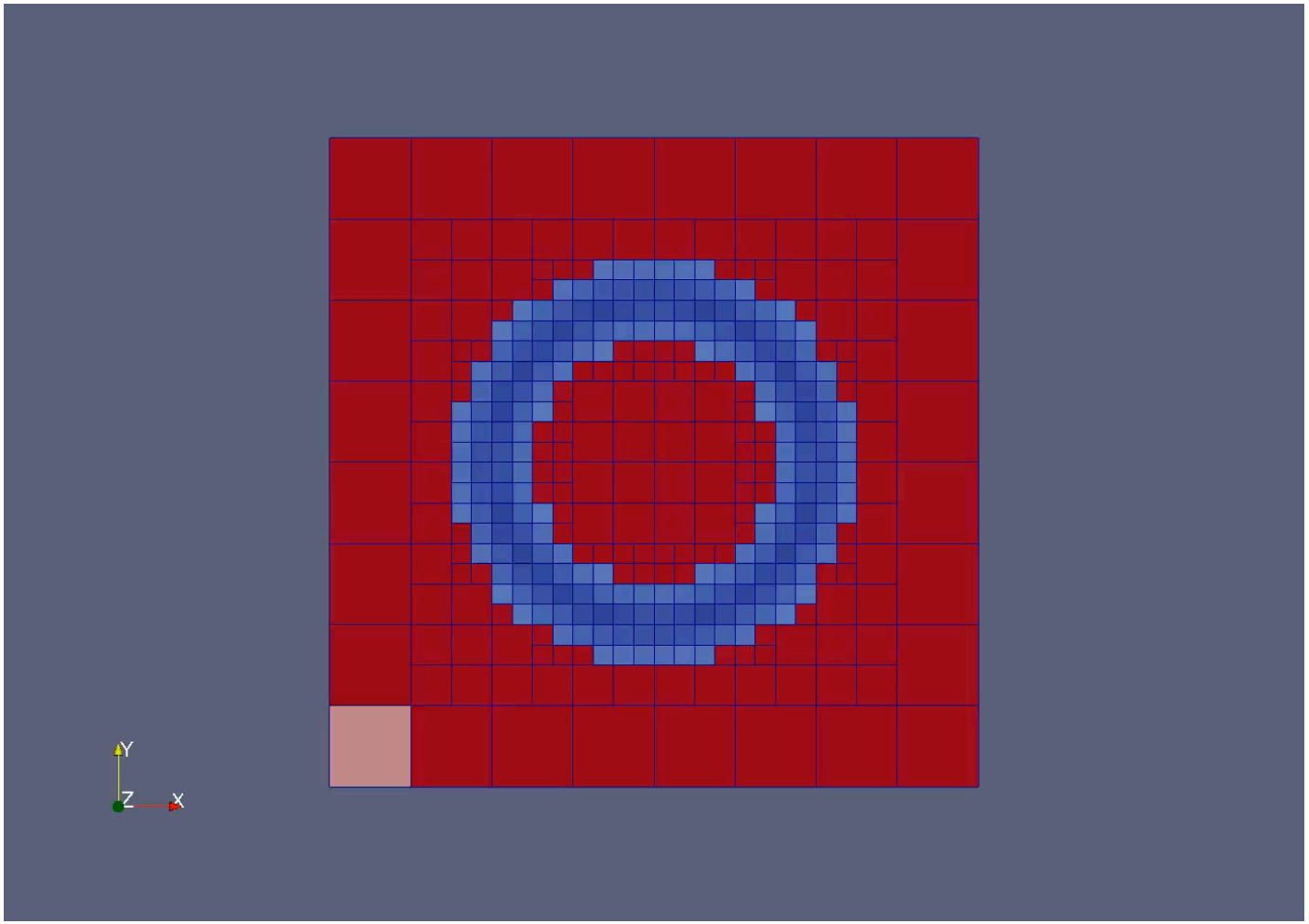


Fig. 1. Case where neighboring cells are smaller than A.



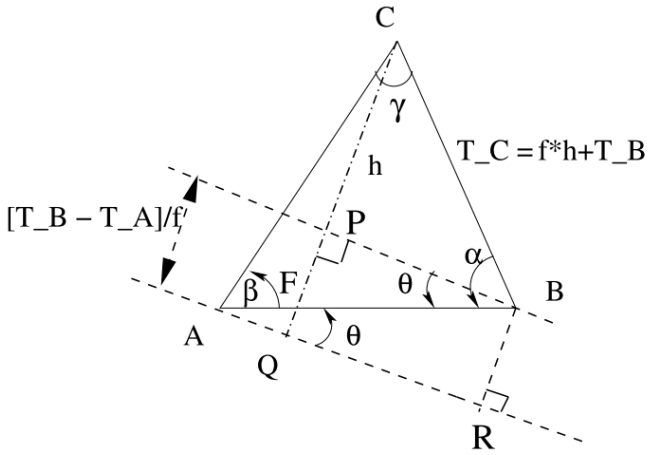


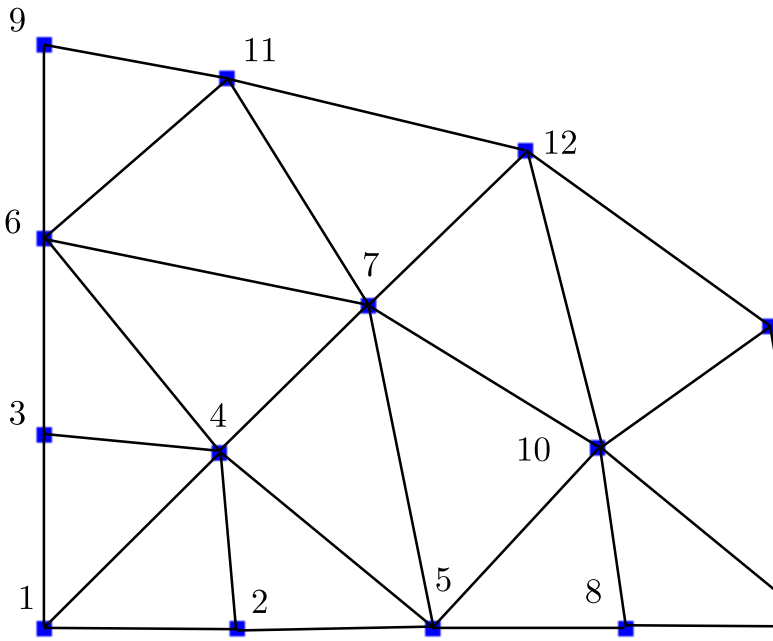
T. C. Cecil, S. J. Osher, and J. Qian, *Simplex free adaptive tree fast evolution methods for solving level set equations in arbitrary dimensi Computational Physics* **213** (2006), no. 2 458–473.


Y
Z
X

Fast sweeping on nonuniform grids

- Discrete equation at each point
- Specific sweeping order

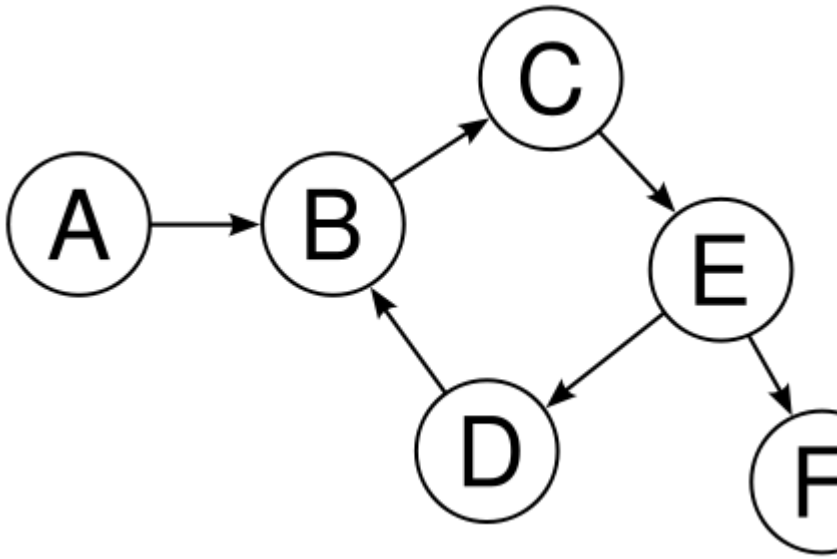




J. Qian, Y. Zhang, and H. Zhao, *Fast Sweeping Methods for Eikonal*
Triangular Meshes, *SIAM Journal on Numerical Analysis* **45** (Jan., 2

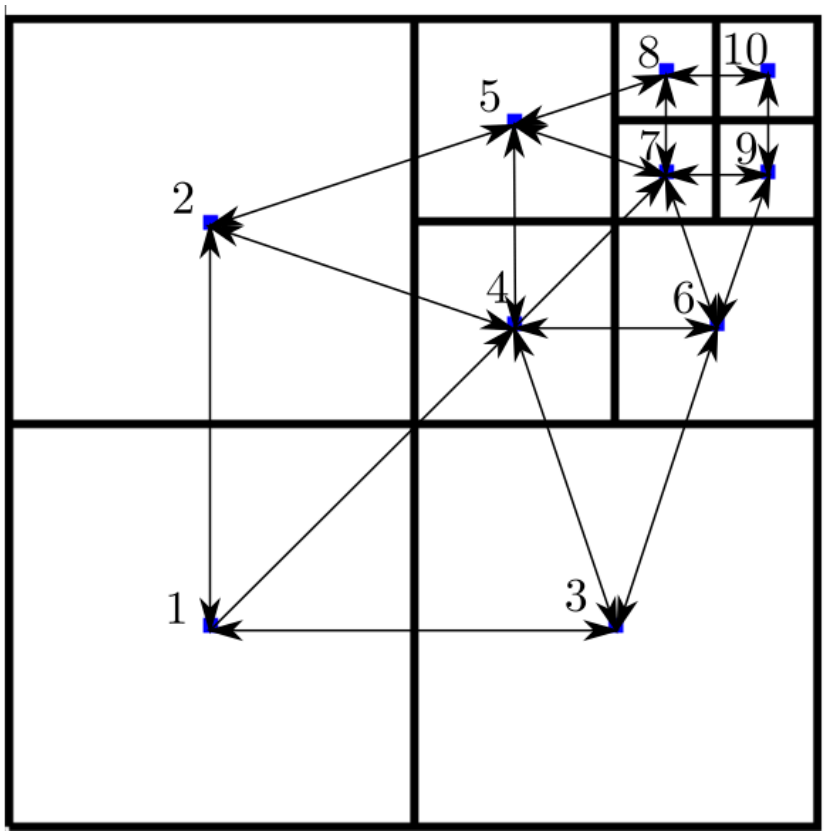
Graph theory interpretation

- A graph represents a set of objects and their connections
  - Vertices
  - Edges
    - Sometimes directed

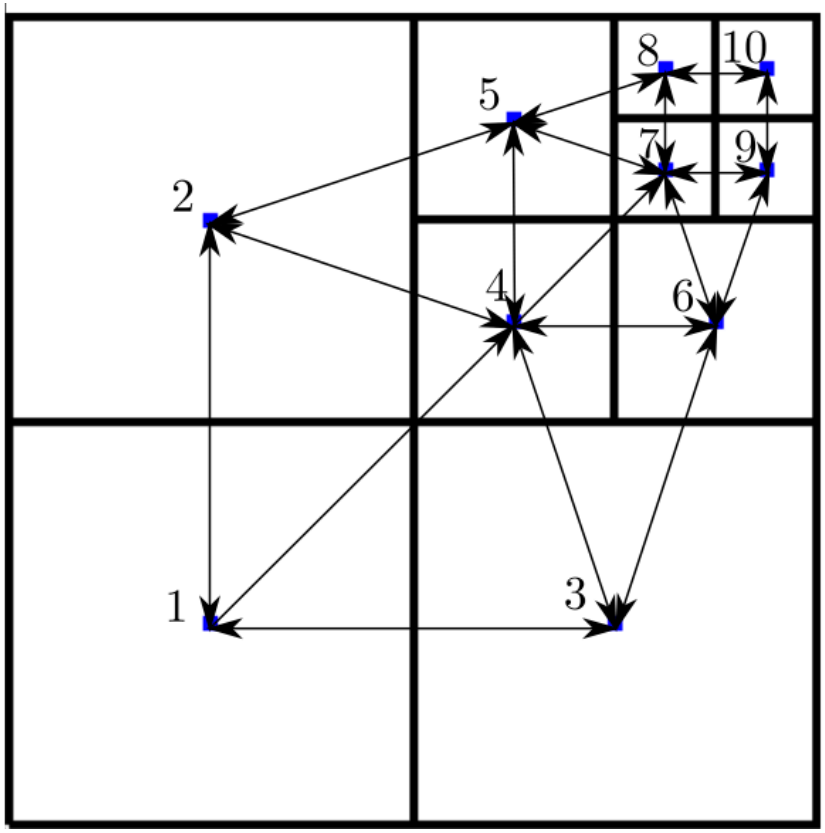


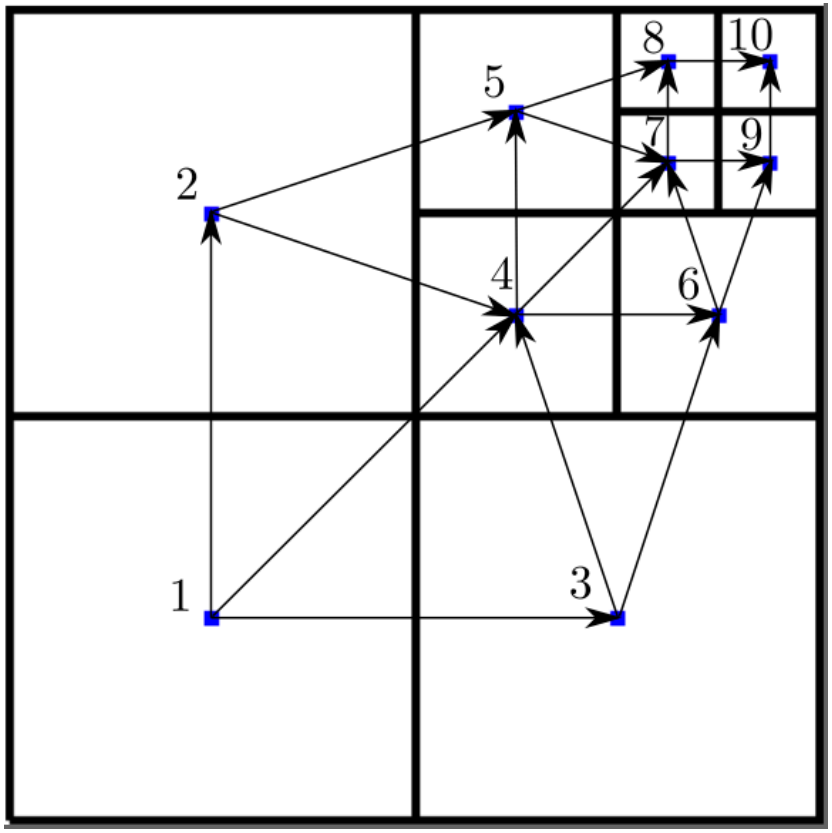
Graph of a quadtree grid

- Vertices are grid points
- Edges represent discretization
- Gives picture of information flow



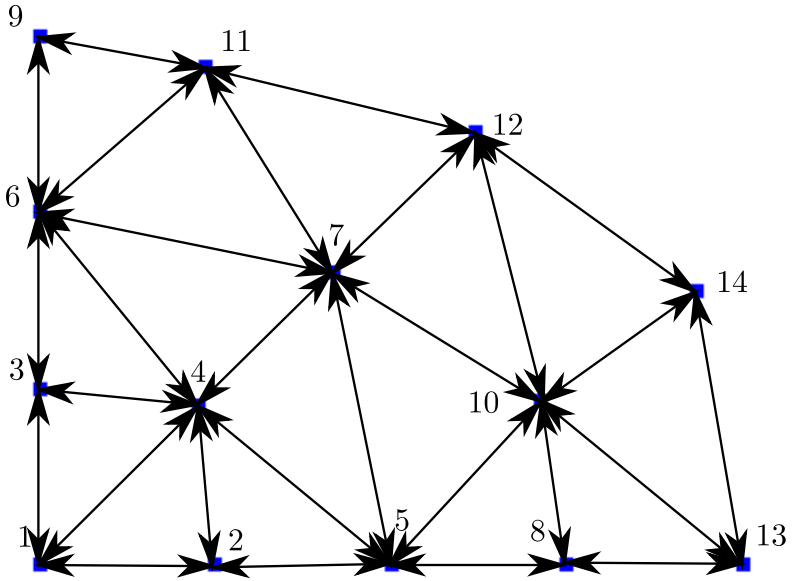
Sweep graph of a quadtree grid

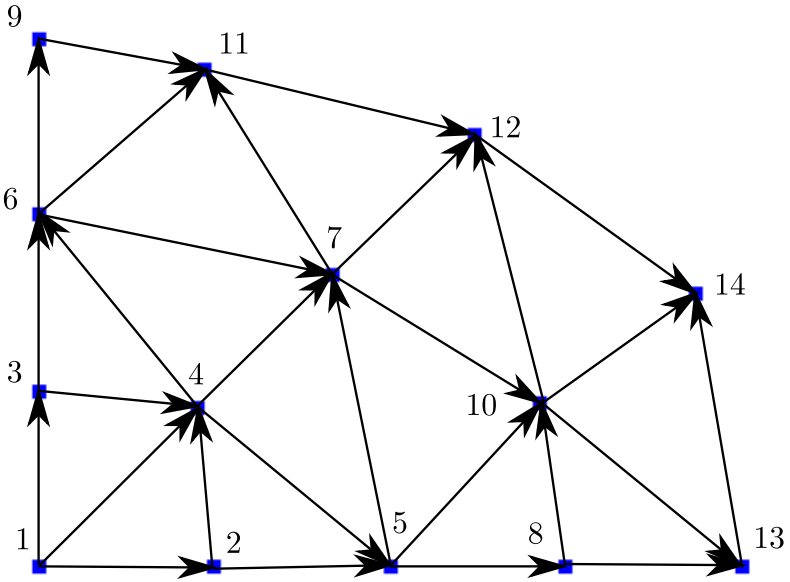




Sweep graph of a triangulated mesh

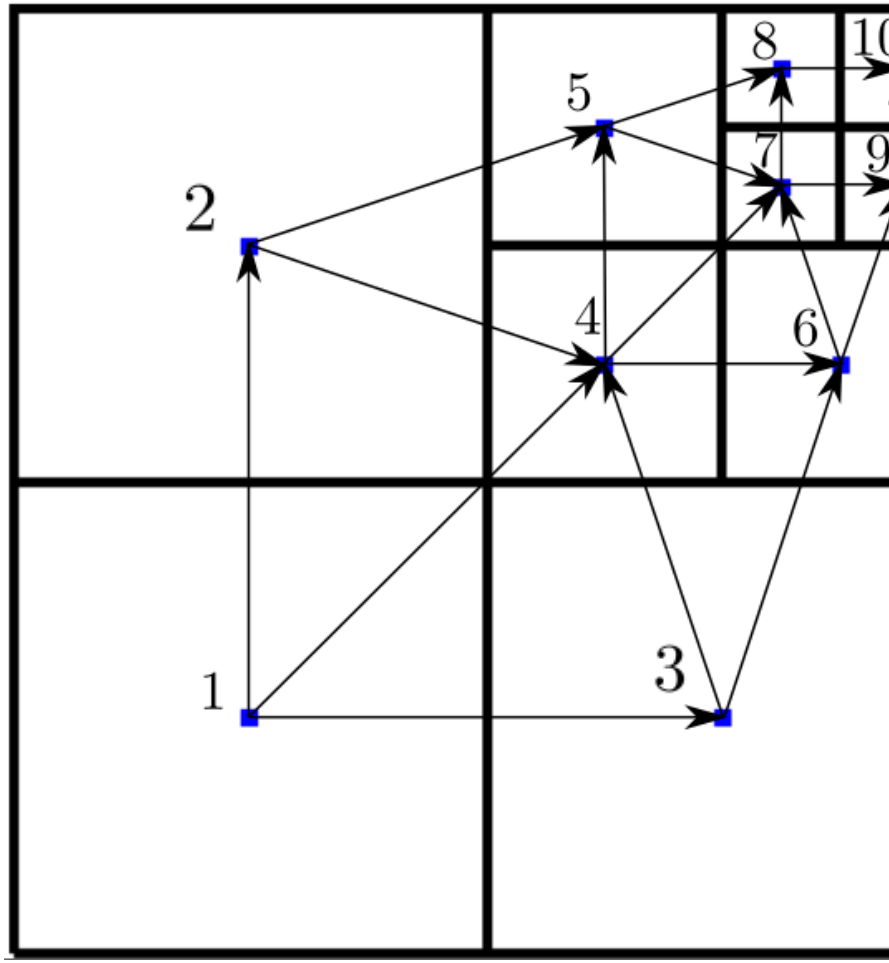
- Sweep graph general to ANY grid





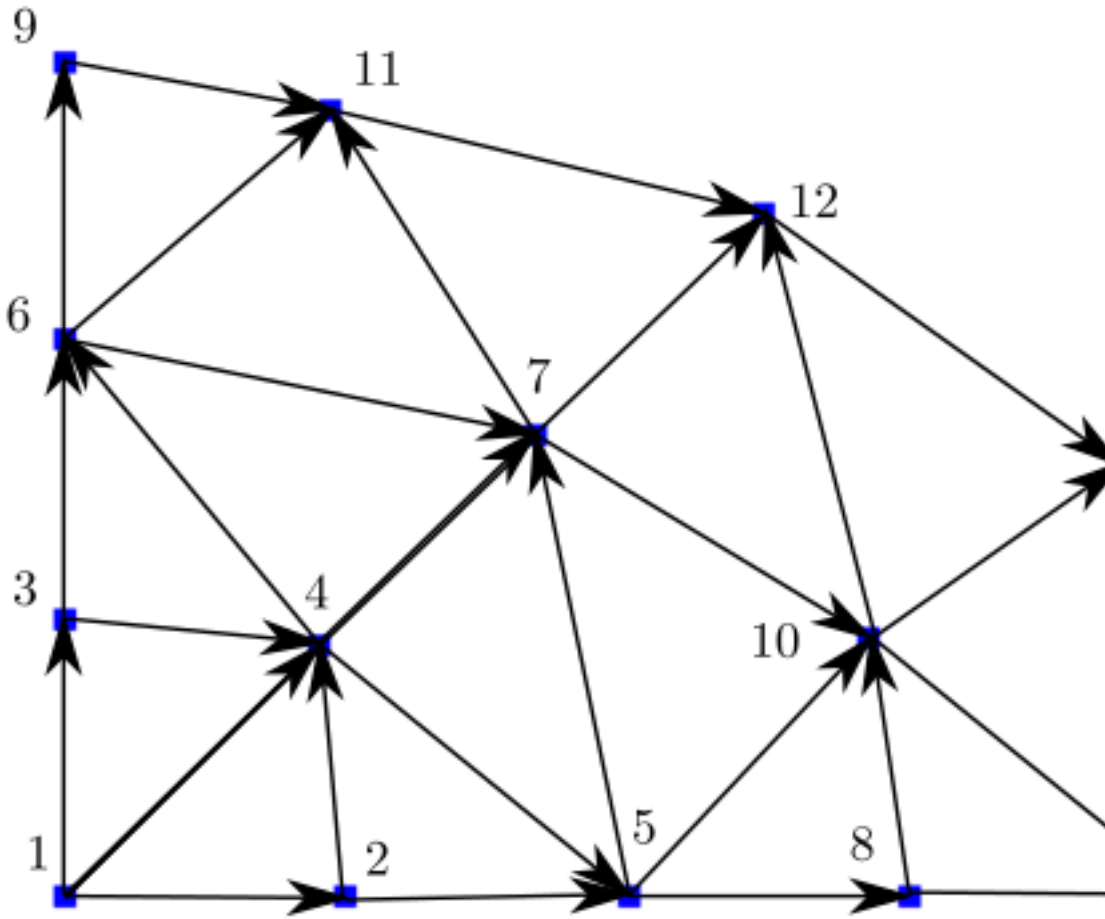
Topological ordering

- An ordering such that vertices only depend on lower numbered vertices
- Morton ordering is one example
- Not unique
- Any grid points with identical "longest paths" can be computed in parallel.



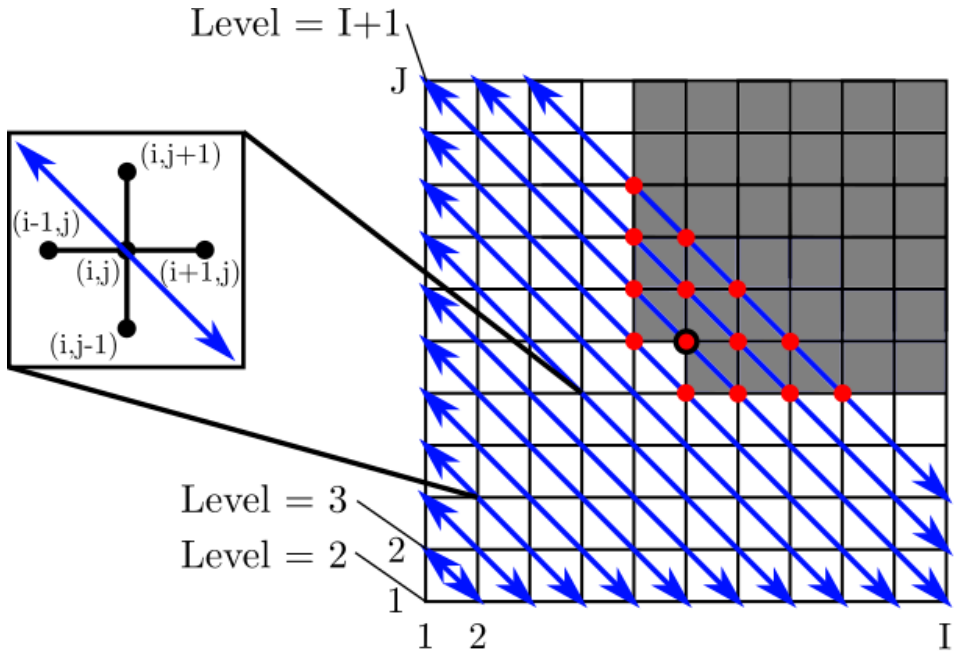
Paths on graphs

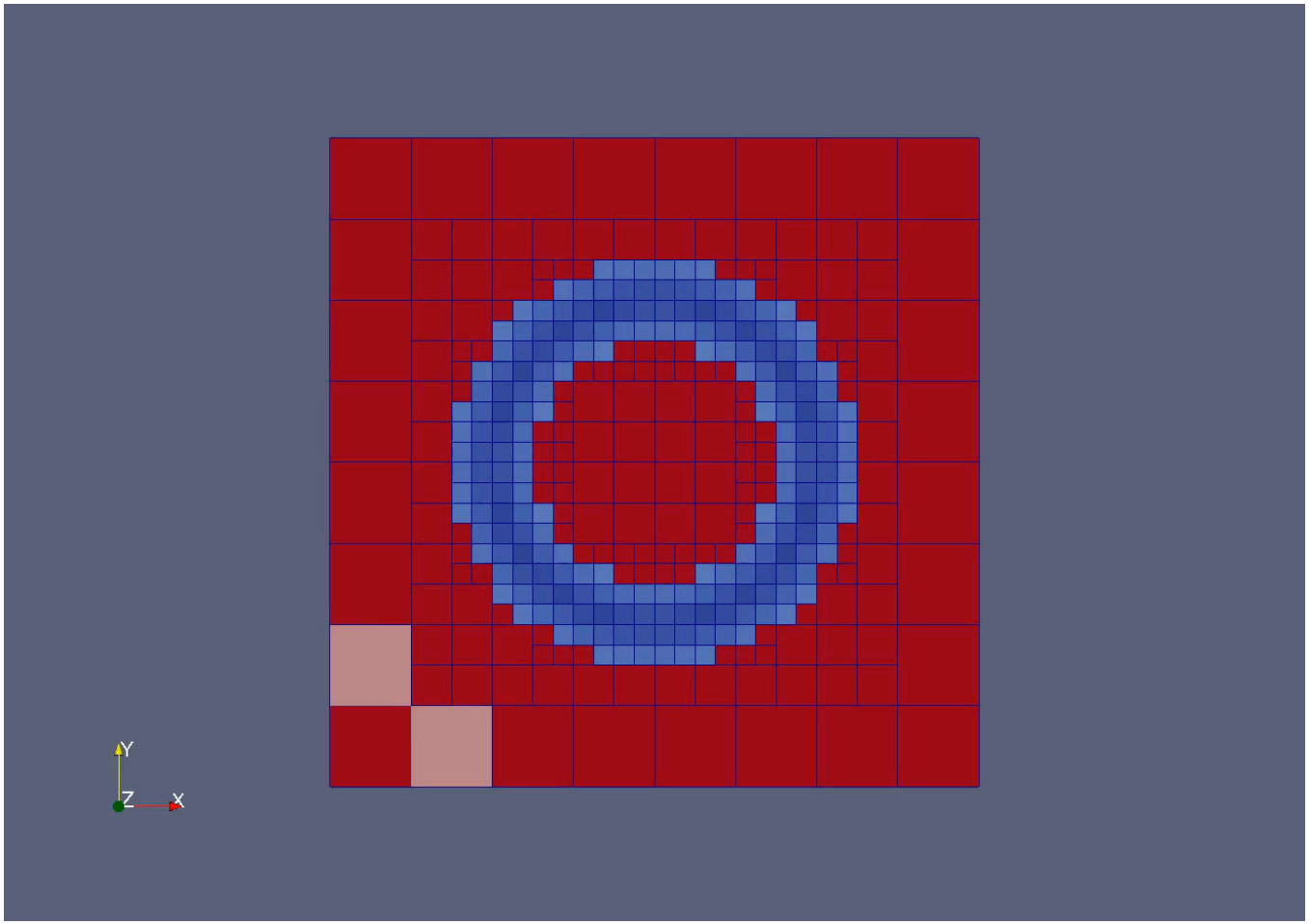
- Shortest path
  - Easy to compute
  - O(V+E)
- Longest path
  - Difficult for general graph
  - Easy for a DAG
  - O(V+E)

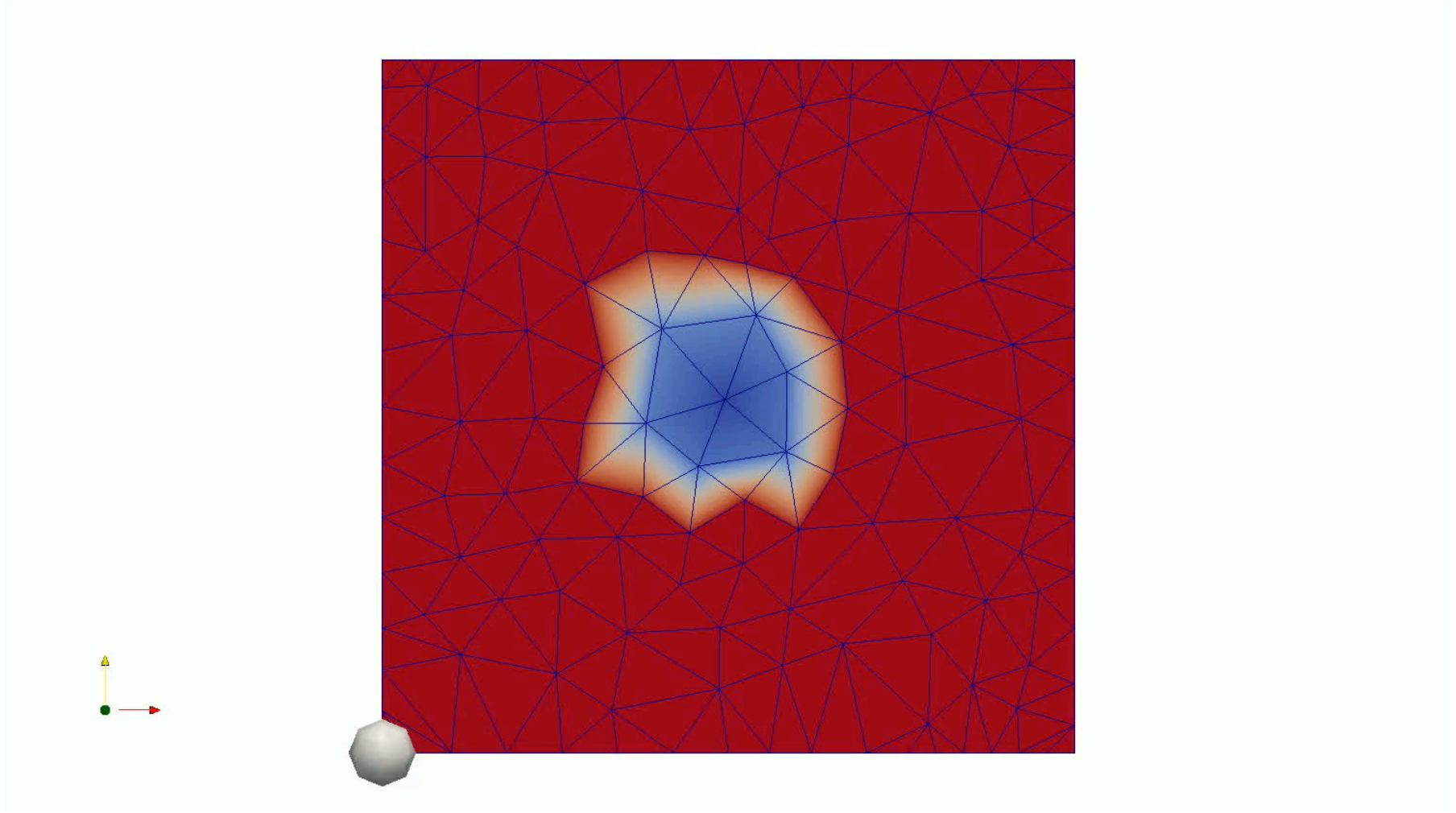


Flashback to uniform grids

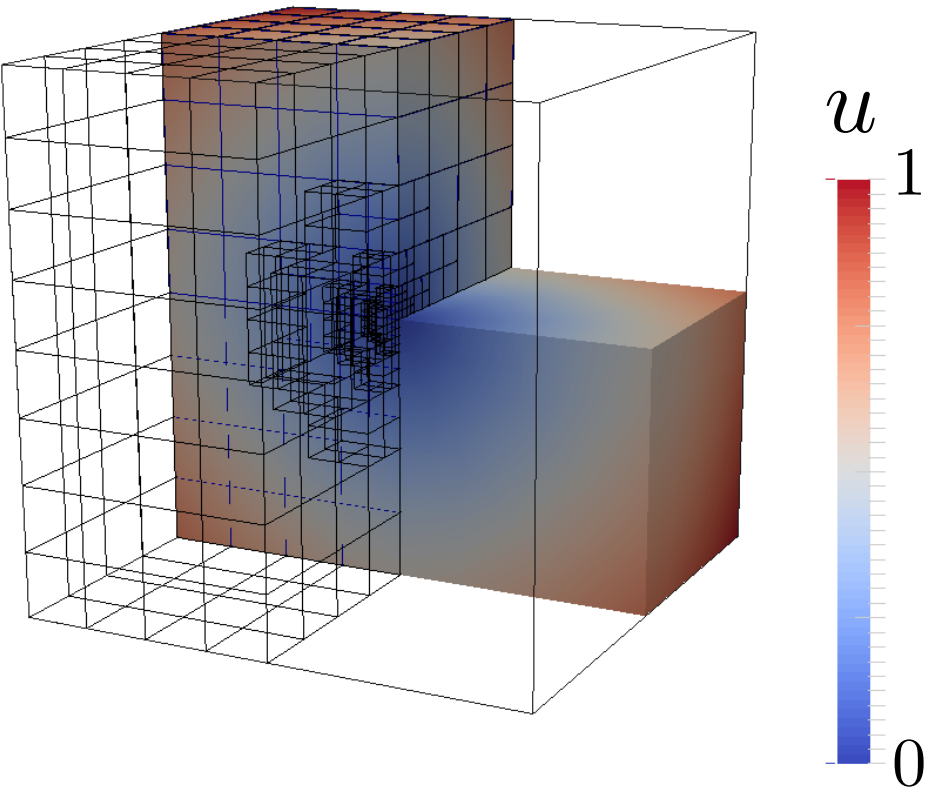
- The levels have identical longest path length!
- The general method exactly reproduces the uniform method

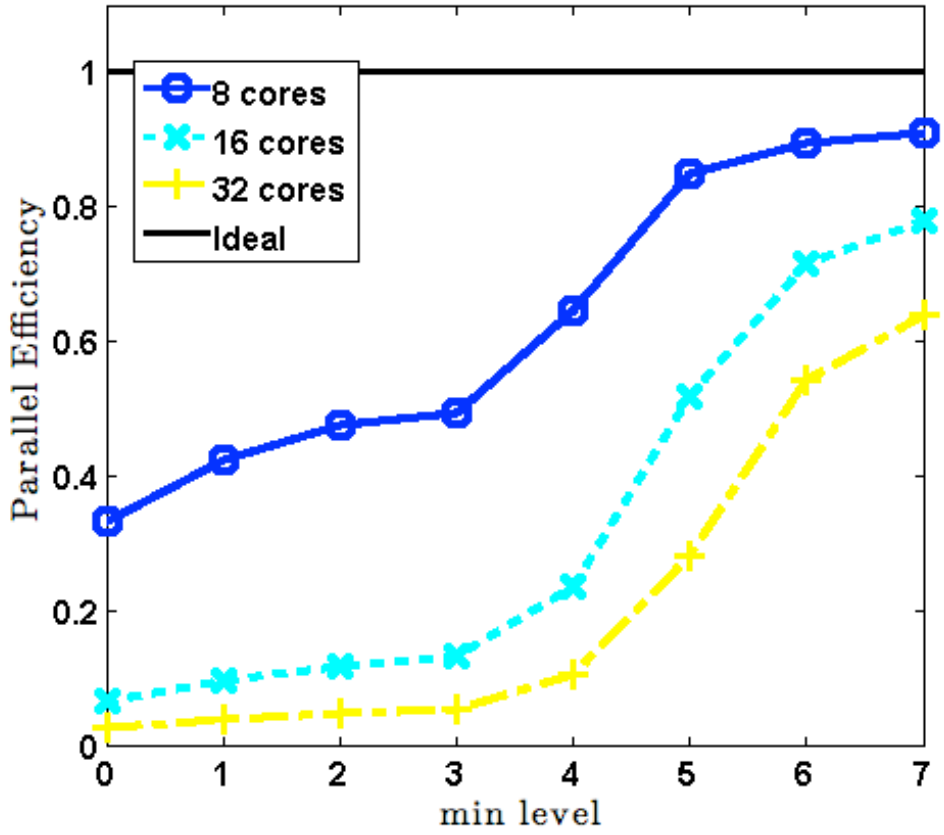



Y
Z
X



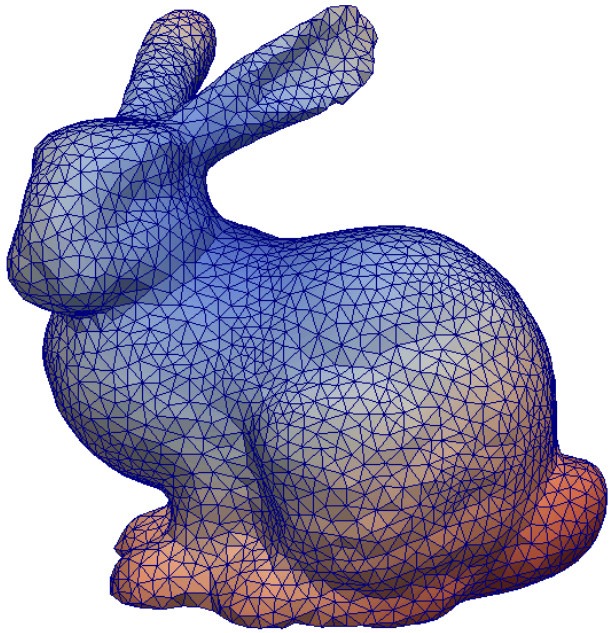
Parallel results

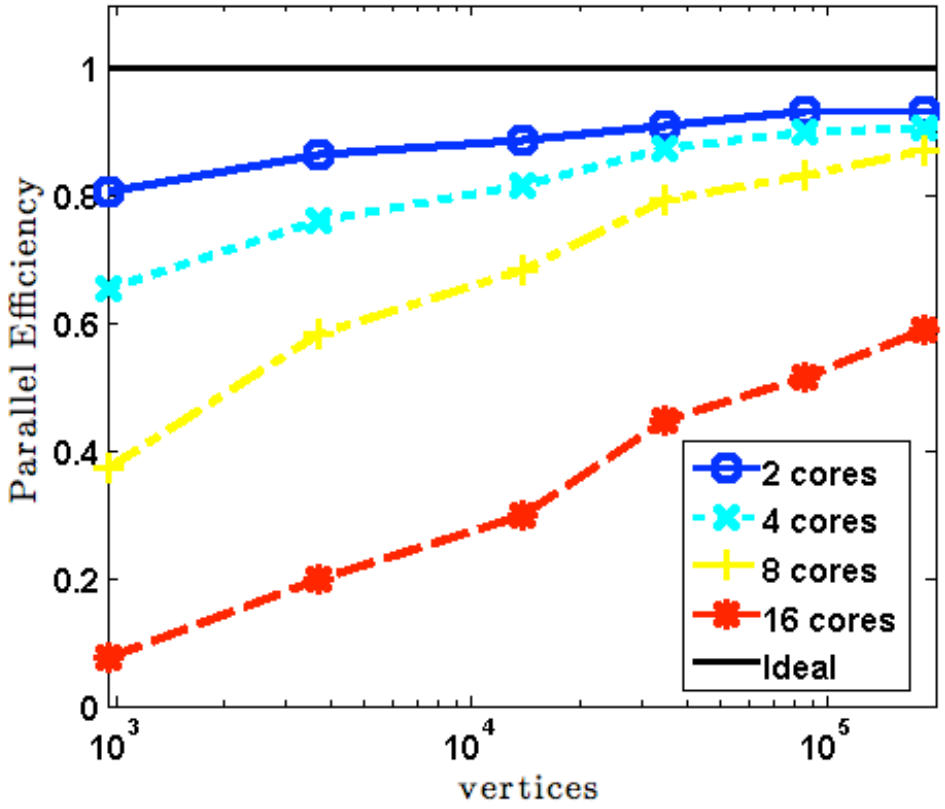




Miles Detrixhe, Frederic Gibou. A parallel fast sweeping method for arbitrary grids (In preparation)

Parallel results





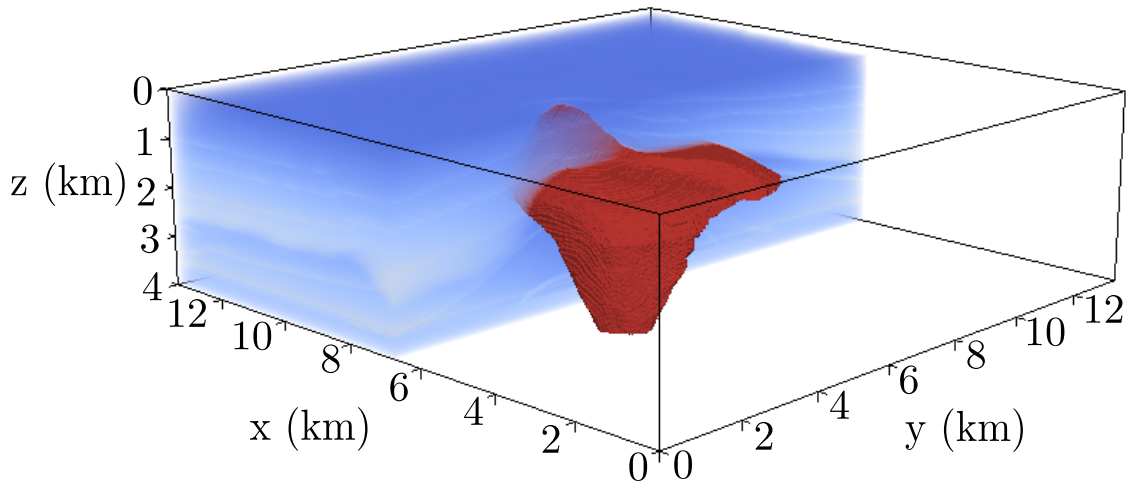
Miles Detrixhe, Frederic Gibou. A parallel fast sweeping method for arbitrary grids (In preparation)

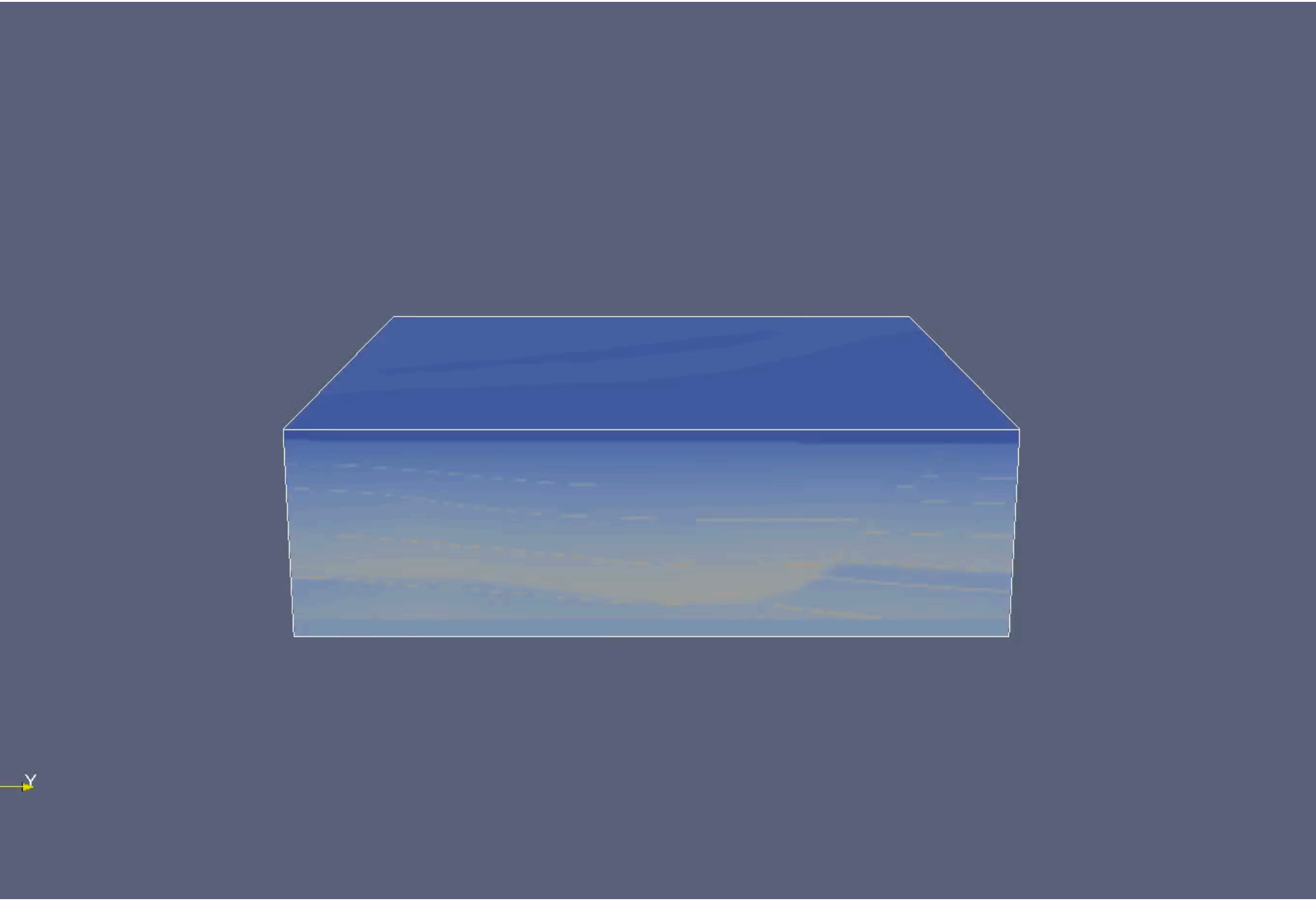
One minute problem

- Seismic wave propagation
- Salt dome velocity model

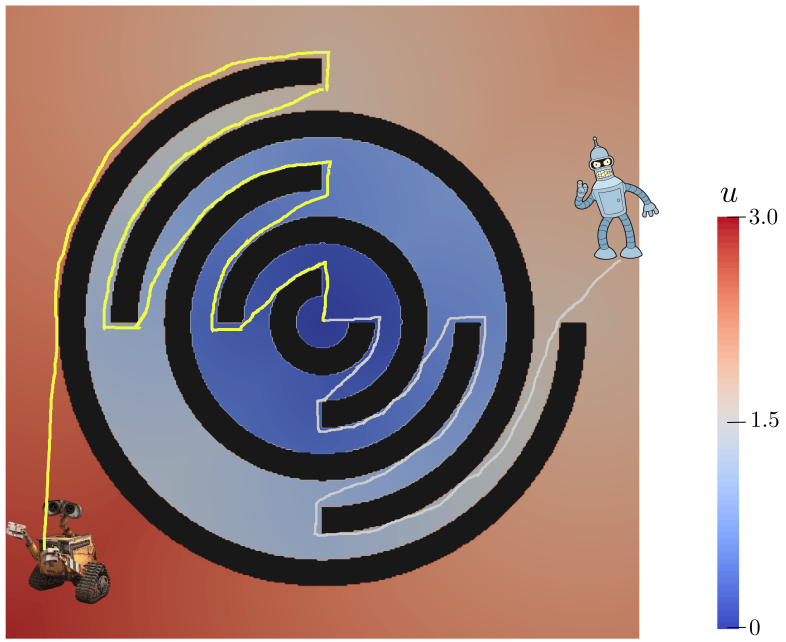
$$
\begin{aligned}
|\nabla u(\mathbf{x})| &= f(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega \subset \mathbb{R}^n, \\
u(\mathbf{x}) &= g(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma \subset \Omega,
\end{aligned}
$$

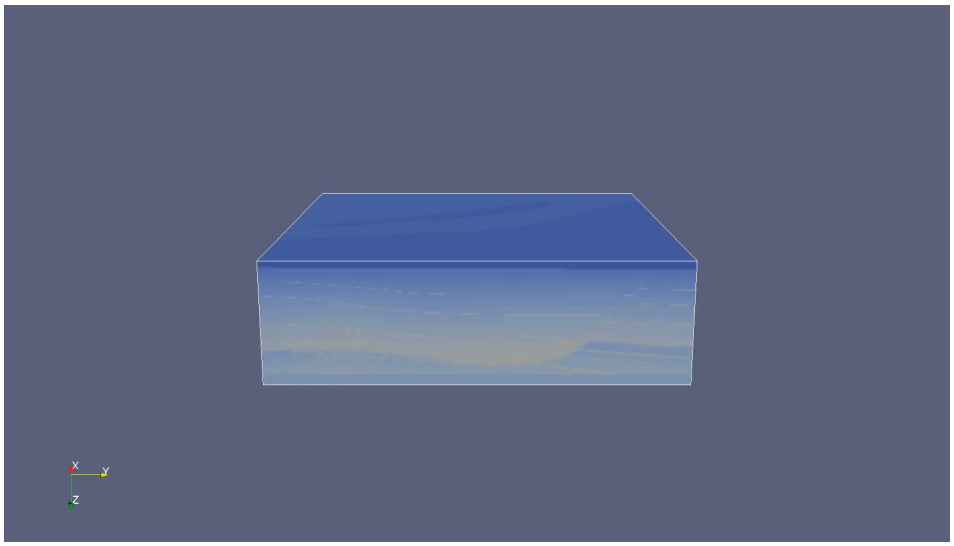
Velocity




Y

One minute problem comparison





Outline

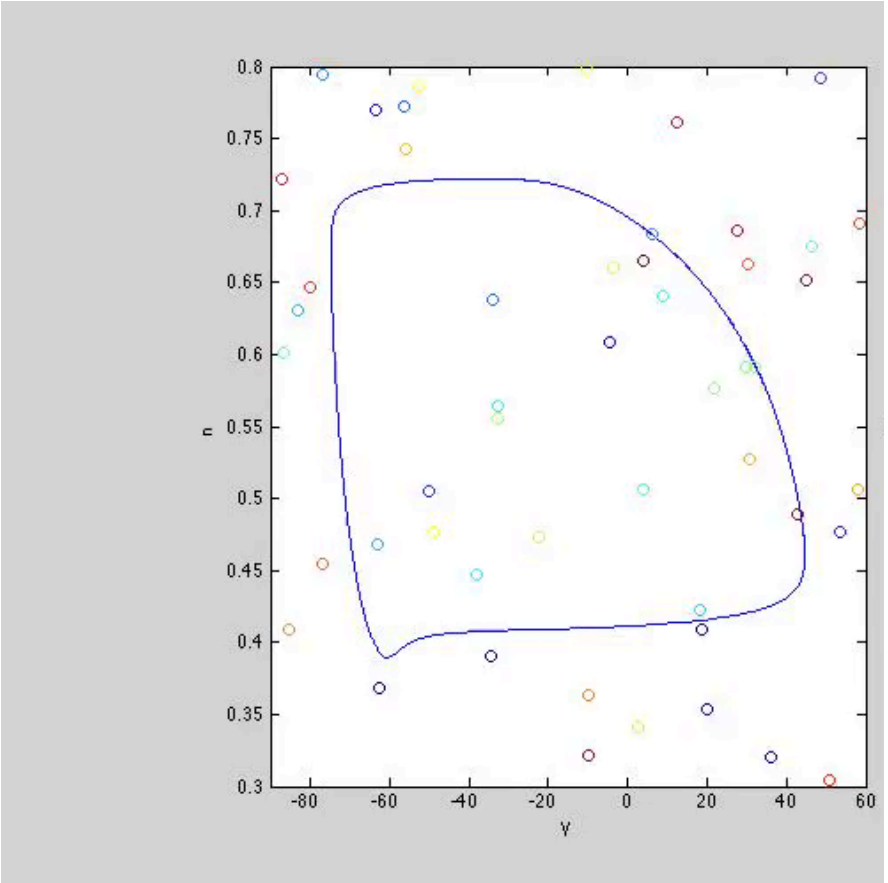
- The fast sweeping method (FSM)
- Parallel computing models
- Parallel fast sweeping on uniform grids
- Hybrid massively parallel algorithm
- Fast sweeping on nonuniform grids
- **Application: computation of isochrons**
- Questions

Isochrons

- Consider a dynamical system

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n,$$

  - With a periodic orbit

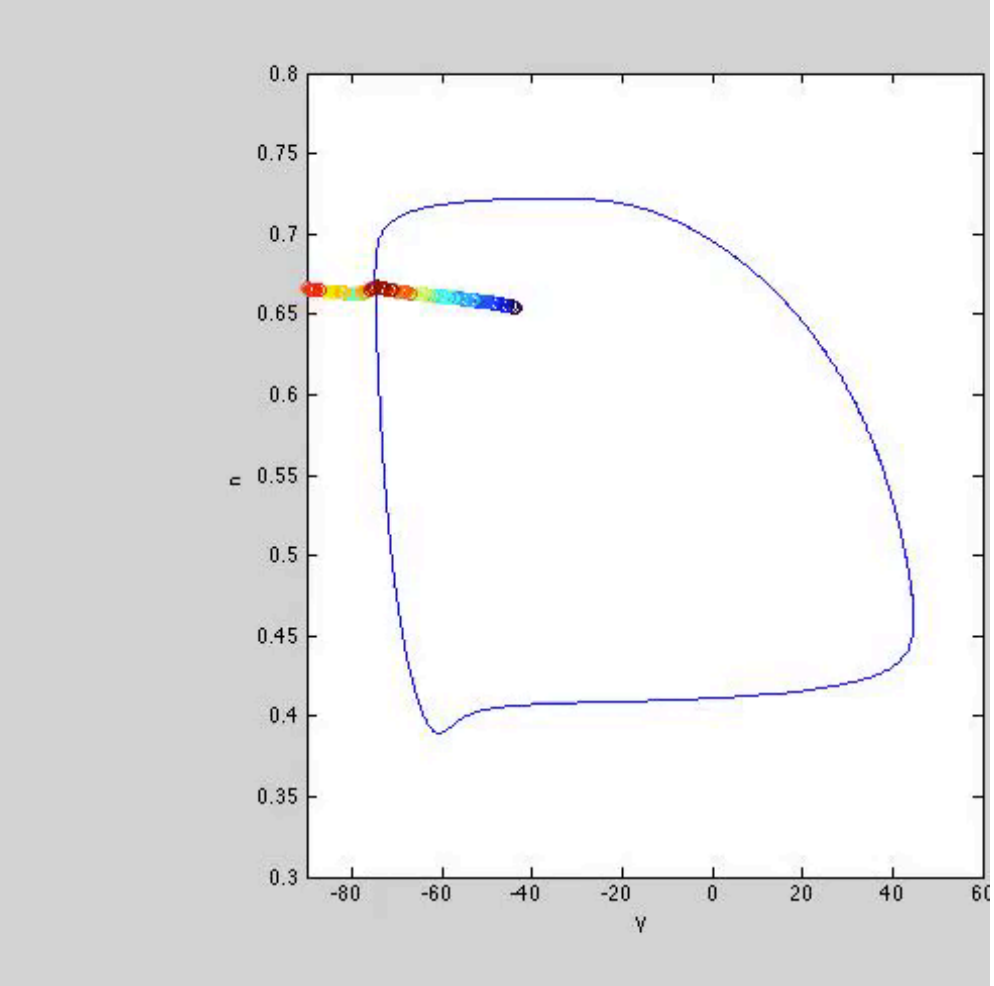


Isochrons

- Consider a dynamical system

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n,$$

  - With a periodic orbit
- All points on an isochron approach the orbit with the same phase



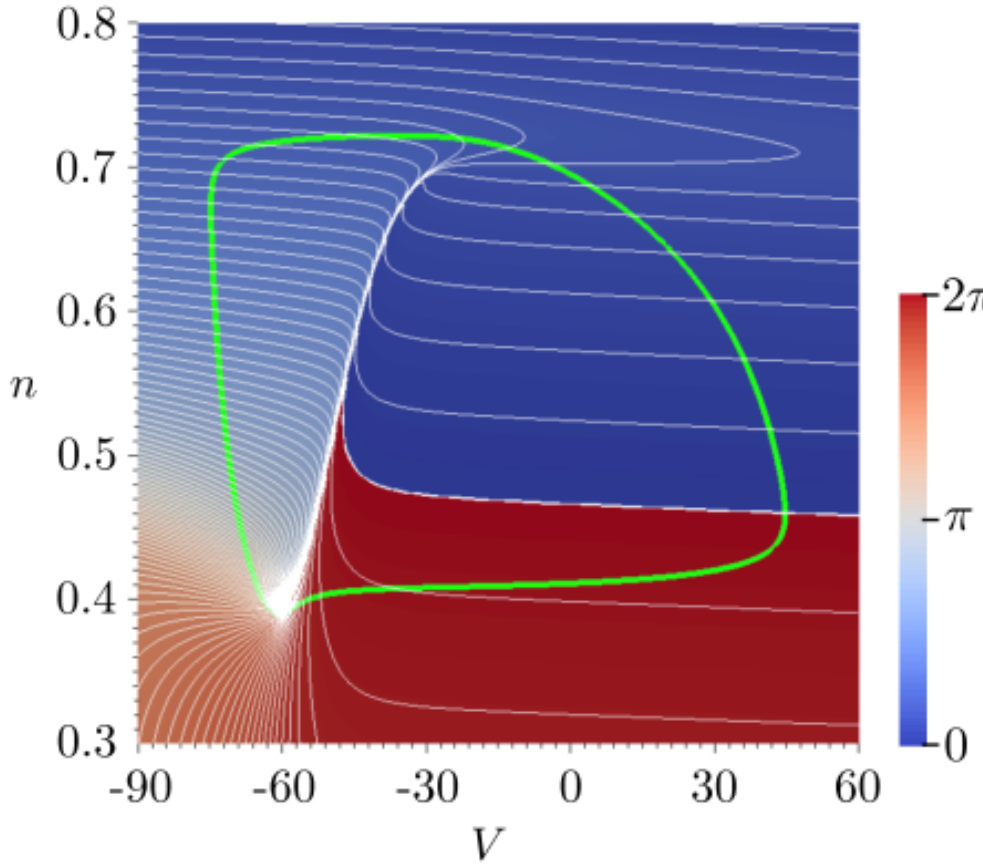
Hodgkin-Huxley model

- Models action potential in giant squid axon
- Used as a simple neuron model

$$C\frac{dV}{dt} = I_b + I - \bar{g}_{Na}[m_\infty(V)]^3(0.8-n)(V-V_{Na})$$

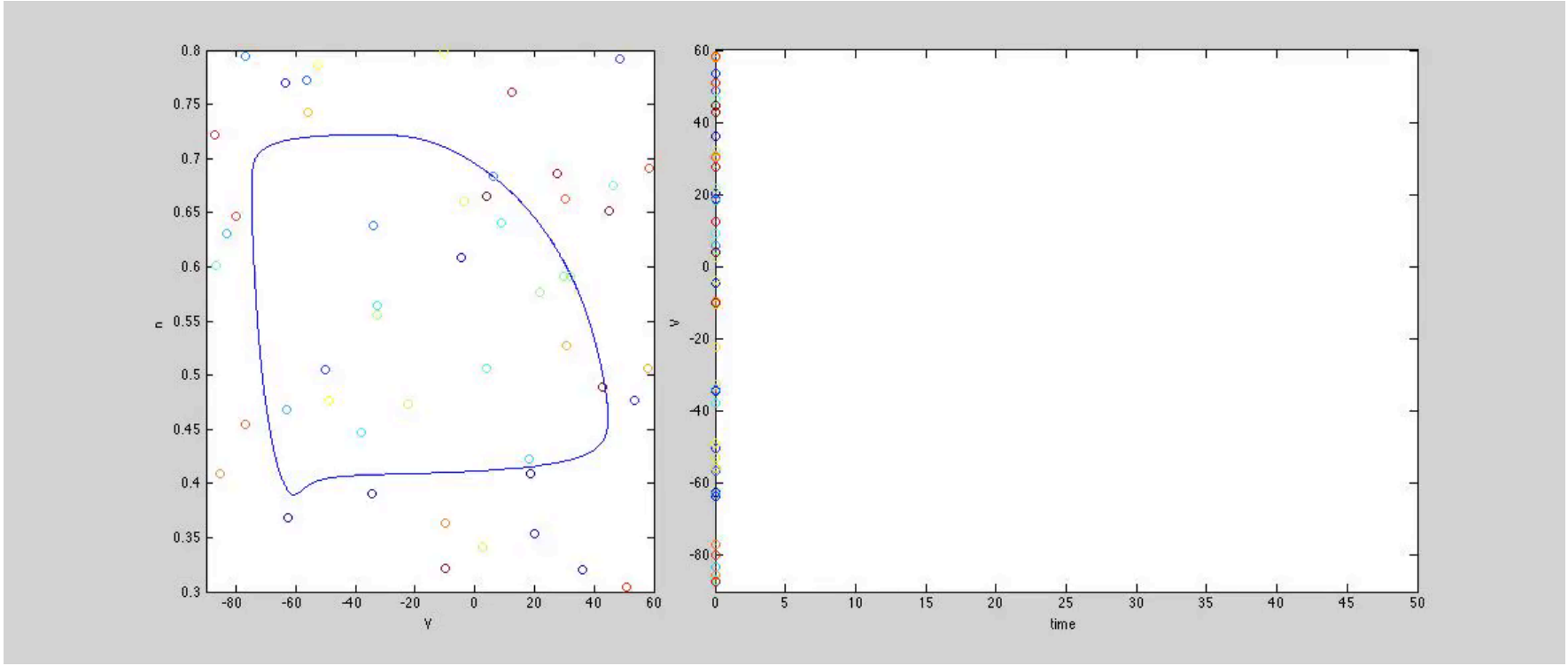
$$-\bar{g}_K n^4(V-V_K) - g_L(V-V_L),$$

$$\frac{dn}{dt} = \alpha_n(V)(1-n) - \beta_n(V)n.$$

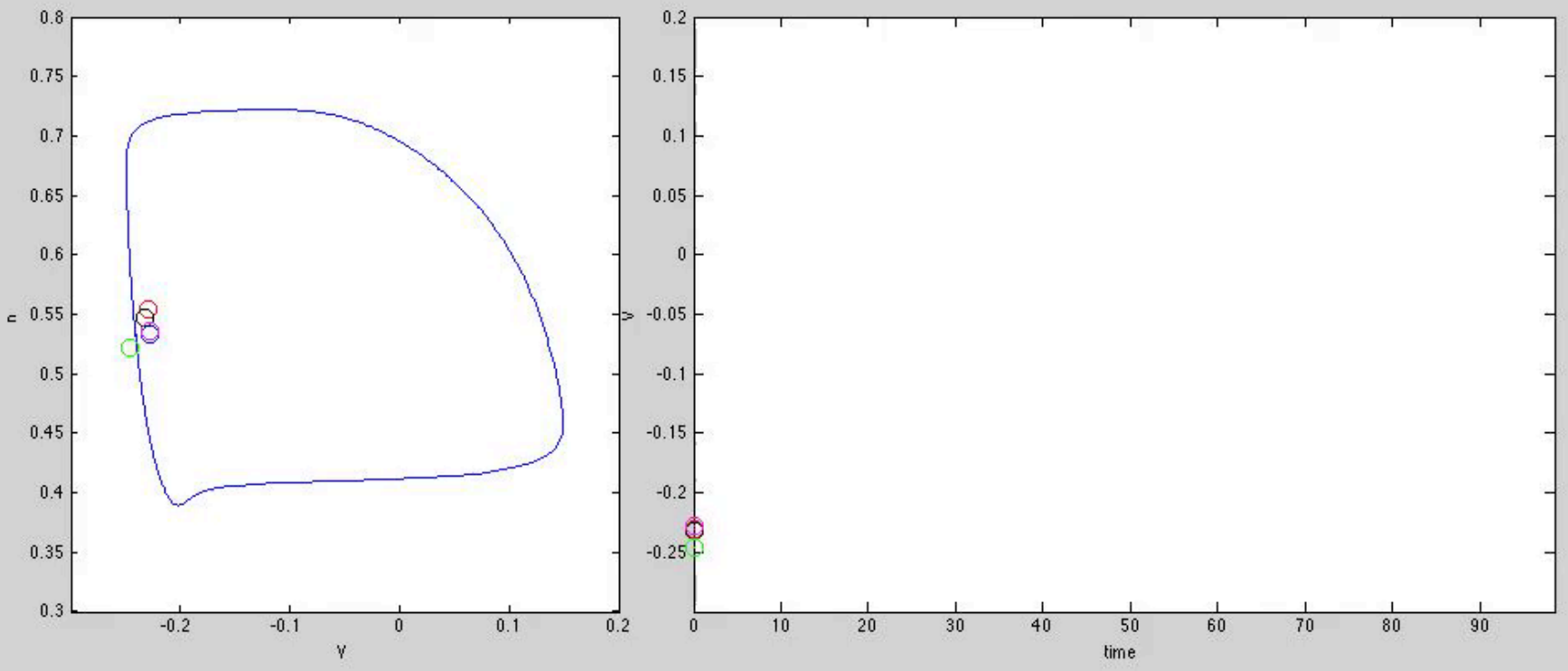


A. Hodgkin and A. Huxley, *A Quantitative Description of Membrane Current its Application to Conduction and Excitation in Nerve*, *The Journal of Physio* (1952) 500–544.

Link to Parkinson's disease treatment



Link to Parkinson's disease treatment



Link to Parkinson's disease treatment

A. Nabi, M. Mirzadeh, F. Gibou, and J. Moehlis, *Minimum Energy Desynchronizing Control for Coupled Neurons.*, *Journal of Computational Neuroscience* **34** (Apr., 2013) 259–71.

Computation of isochrons

- Definition of phase

$$\frac{d\theta}{dt} = \frac{2\pi}{T}$$

$$\nabla\theta \cdot \mathbf{F}(\mathbf{x}) = \frac{2\pi}{T}.$$

- Chain rule

$$\frac{d\theta}{dt} = \frac{\partial\theta}{\partial\mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} = \frac{2\pi}{T},$$

Computation of isochrons

- Definition of phase

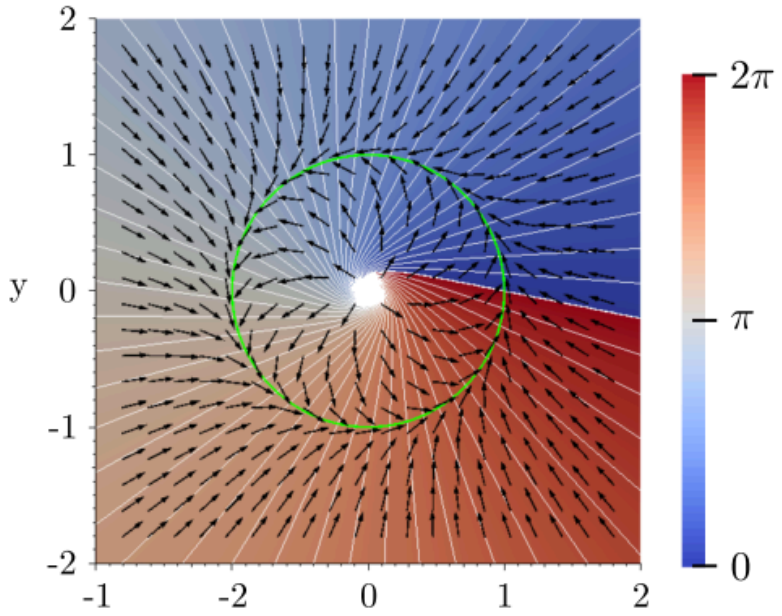
$$\frac{d\theta}{dt} = \frac{2\pi}{T}$$

$$\begin{aligned} \nabla\theta \cdot \mathbf{F}(\mathbf{x}) &= \frac{2\pi}{T} \quad \forall\, \mathbf{x} \in \mathcal{B}, \\ \theta(\gamma) &= f(\gamma) \quad \forall\, \gamma \in \Gamma^{+}, \end{aligned}$$

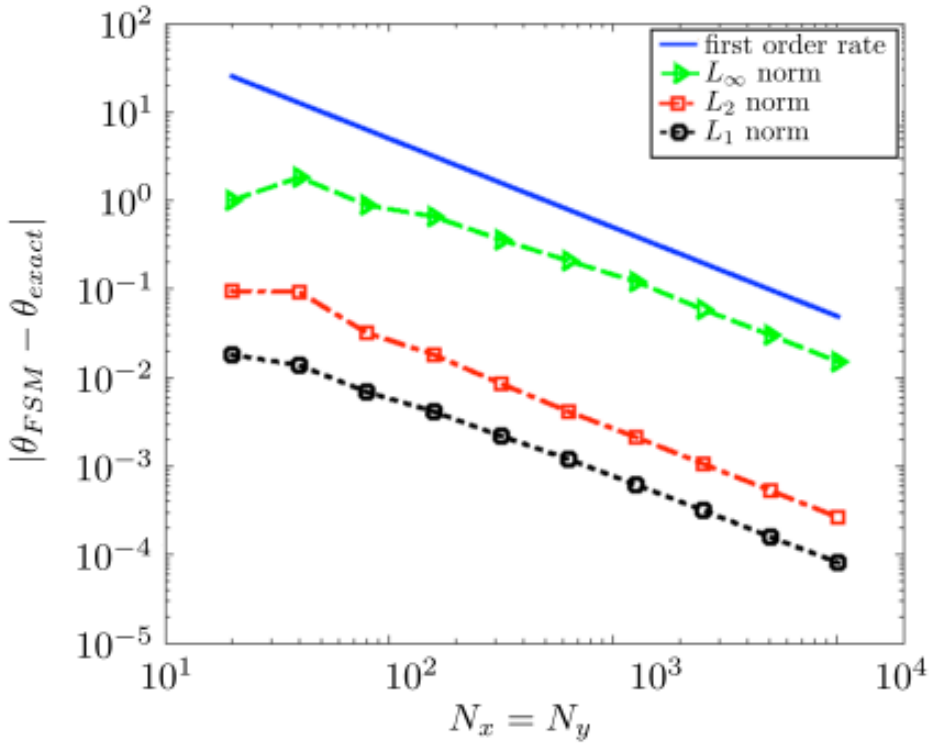
- Chain rule

$$\frac{d\theta}{dt} = \frac{\partial\theta}{\partial\mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} = \frac{2\pi}{T},$$

Validation



$$\dot{} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 5r^2(1-r)\cos\phi - r^2\sin\phi \\ 5r^2(1-r)\sin\phi + r^2\cos\phi \end{pmatrix}$$



Miles Detrixhe, Jeff Moehlis, Marion Doubeck, Frederic Gibou. A fast Eulerian approach for computation of global isochrons in high dimensions (first revision).

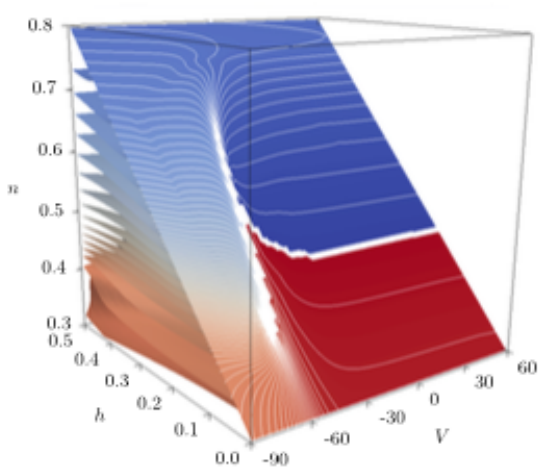
4D Hodgkin-Huxley model

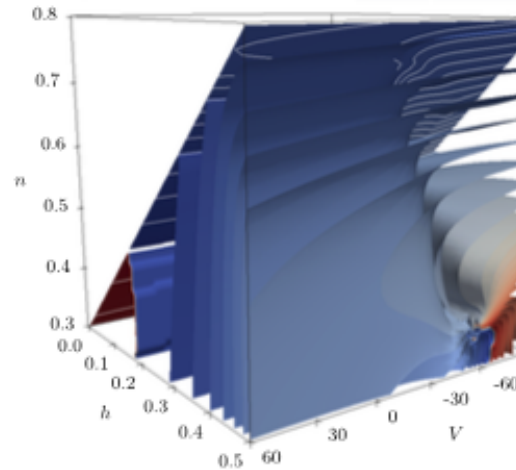
- Better model
- More physics captured

$$
\begin{cases}
C\dfrac{dV}{dt} &= I_b + I - \bar{g}_{Na} m^3 h (V - V_{Na}) - \bar{g}_K n^4 (V - V_K) - g_L (V - V \\
\dfrac{dn}{dt} &= \alpha_n(V)(1-n) - \beta_n(V) n, \\
\dfrac{dm}{dt} &= \alpha_m(V)(1-m) - \beta_m(V) m, \\
\dfrac{dh}{dt} &= \alpha_h(V)(1-h) - \beta_h(V) h,
\end{cases}
$$

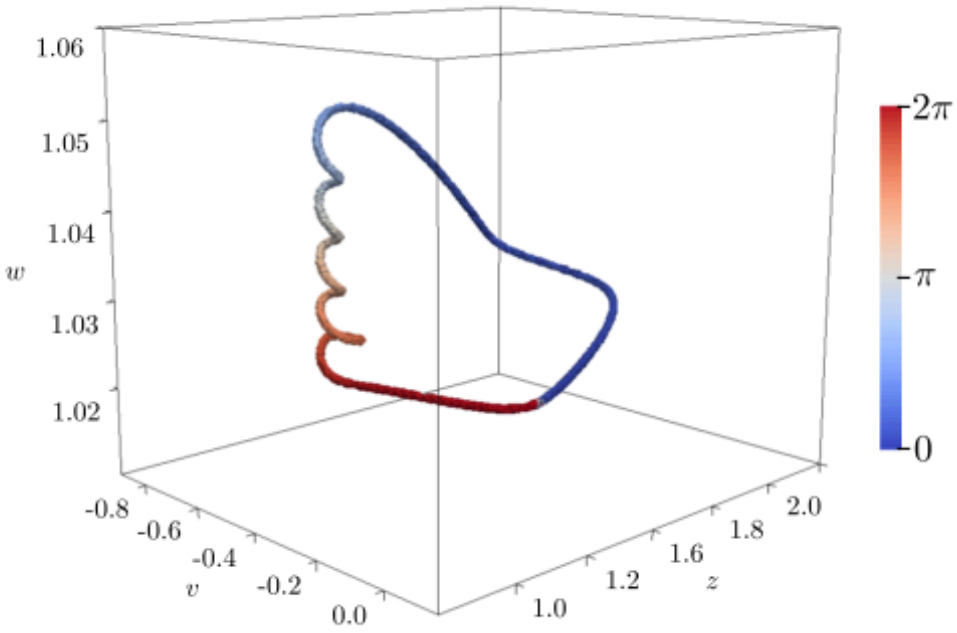
4D Hodgkin-Huxley model

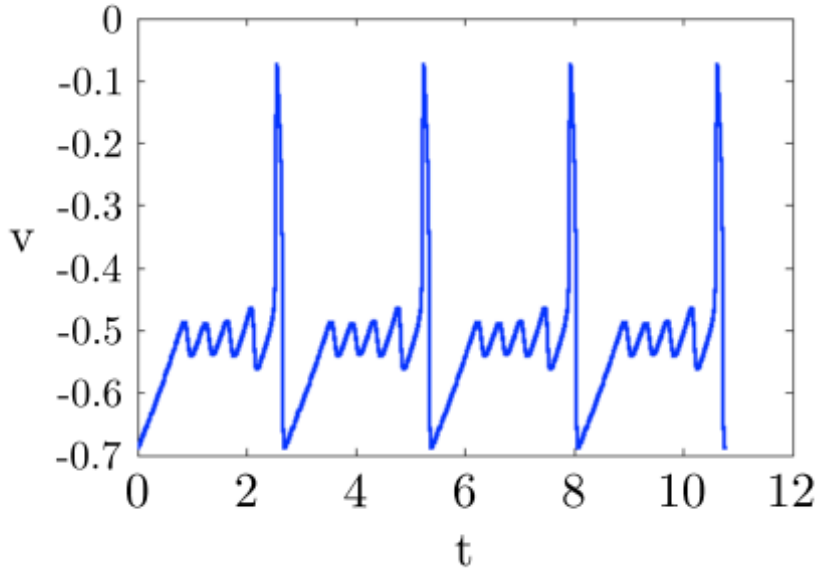
- Better model
- More physics captured
- Better control?





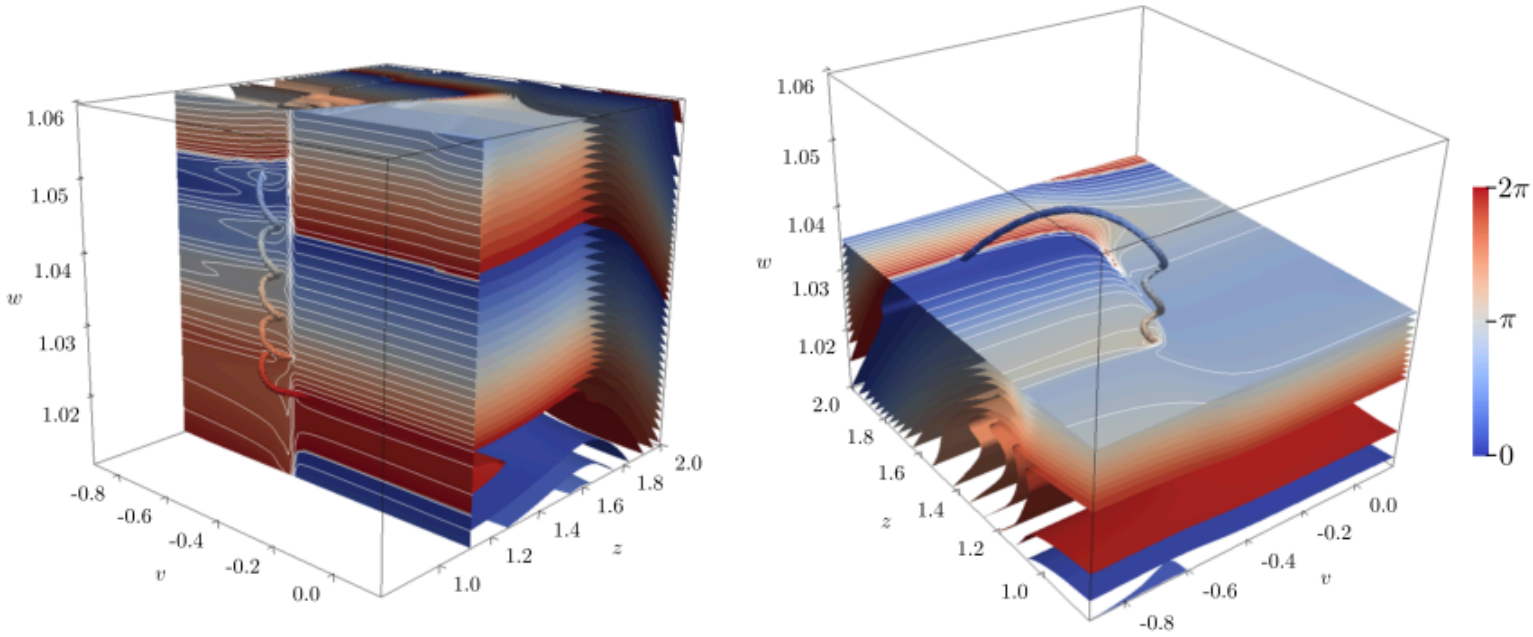
Wilson-Callaway model



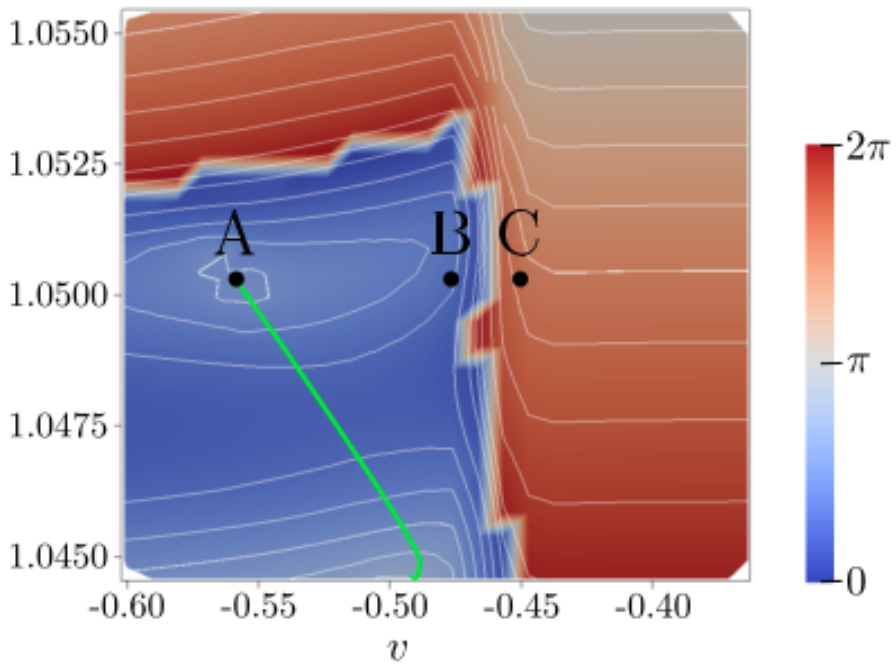


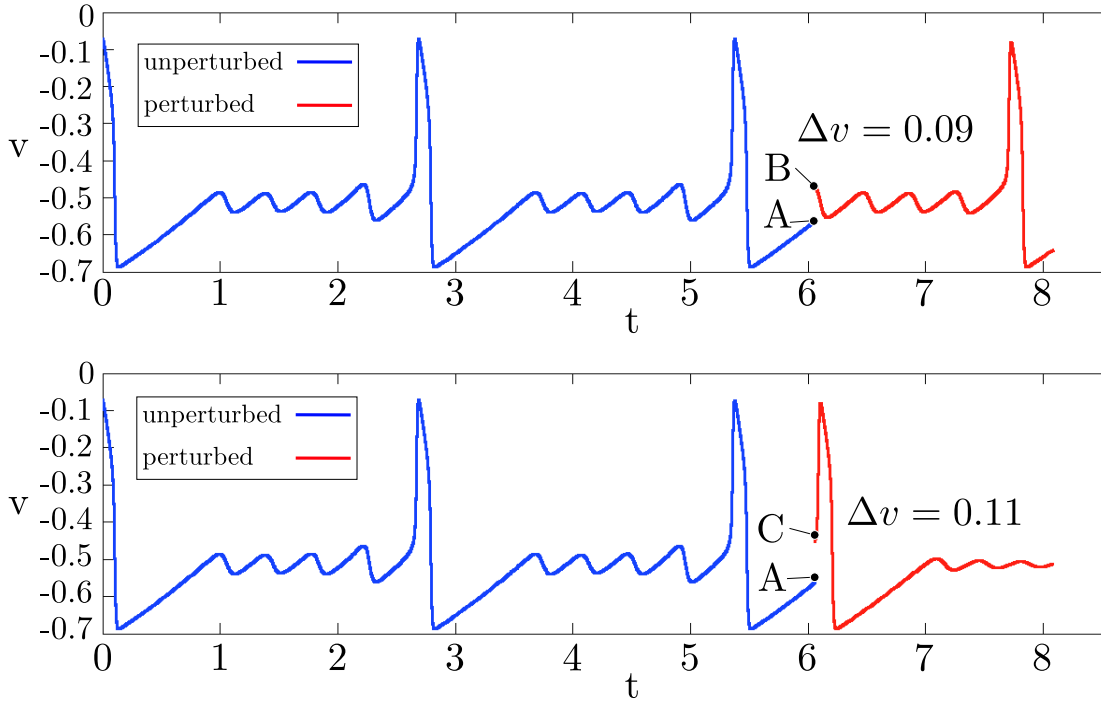
Wilson, C. J., and Callaway, J. C. Coupled Oscillator Model of the Dopaminergic Neuron of the Substantia Nigra. Journal of Neurophysiology 83, 5 (2000), 3084–3100.

Isochrons of Wilson-Callaway



Isochrons of Wilson-Callaway





Summary

- Parallel fast sweeping on uniform grids
- Hybrid massively parallel algorithm
- Fast sweeping on nonuniform grids
- Applications
  - Path planning
  - Distance to a surface
  - Coverage control
  - Dynamic games
  - Seismic wave arrival time
  - Computation of isochrons

Journal articles

- Related to this talk
  - Miles Detrixhe, Frederic Gibou, Chohong Min, A parallel fast sweeping method for the Eikonal equation Journal of Computational Physics, Volume 237, 15 March 2013, Pages 46-55
  - Miles Detrixhe, Frederic Gibou Hybrid massively parallel fast sweeping method for static Hamilton-Jacobi equations. Journal of Computational Physics, Volume 322, 2016, Pages 199-223
  - Miles Detrixhe, Jeff Moehlis, Marion Doubeck, Frederic Gibou. A fast Eulerian approach for computatio of global isochrons in high dimensions. SIAM Journal on Applied Dynamical Systems, Volume 15 Number 3, 2016, Pages 1501-1527
  - Miles Detrixhe, Frederic Gibou. A parallel fast sweeping method for arbitrary grids (In preparation)
- Other
  - Detrixhe, M. and Aslam, Tariq. From level set to volume of fluid and back again at second-order accuracy. Int. J. Numer. Meth. Fluids, 10 August 2015.
  - Detrixhe, M. et. al. (2010). Ultrarelativistic magnetic monopole search with the ANITA-II balloon-borne radio interferometer. Physical Review D 28(3), 83(2), 1-11.

Acknowledgements

- Frederic Gibou
  - PhD advisor
- Jeff Moehlis
  - Isochrons project collaborator
- Tariq Aslam
  - LANL supervisor

Questions



Bonus slides!

Speedup data hybrid method

Example	Grid size	Processors	cores per partition	Execution time	Speedup	Efficiency
2	640^3	1728	8	2.958	237	0.0958
3	80^4	2048	8	0.658	474.7	0.232
5	160^3	2000	16	82.0	1181	0.590
6	160^3	2000	16	328.5	538.5	0.269

Table 2: Summary of speedup data on largest number of processors

2D HH isochron FSM convergence

