

Stabilization of Finite-Element High-Order Methods with Fourier Spectral Filters: Surviving High-Re, High-Ma flows, and Coarse Grids

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Outline

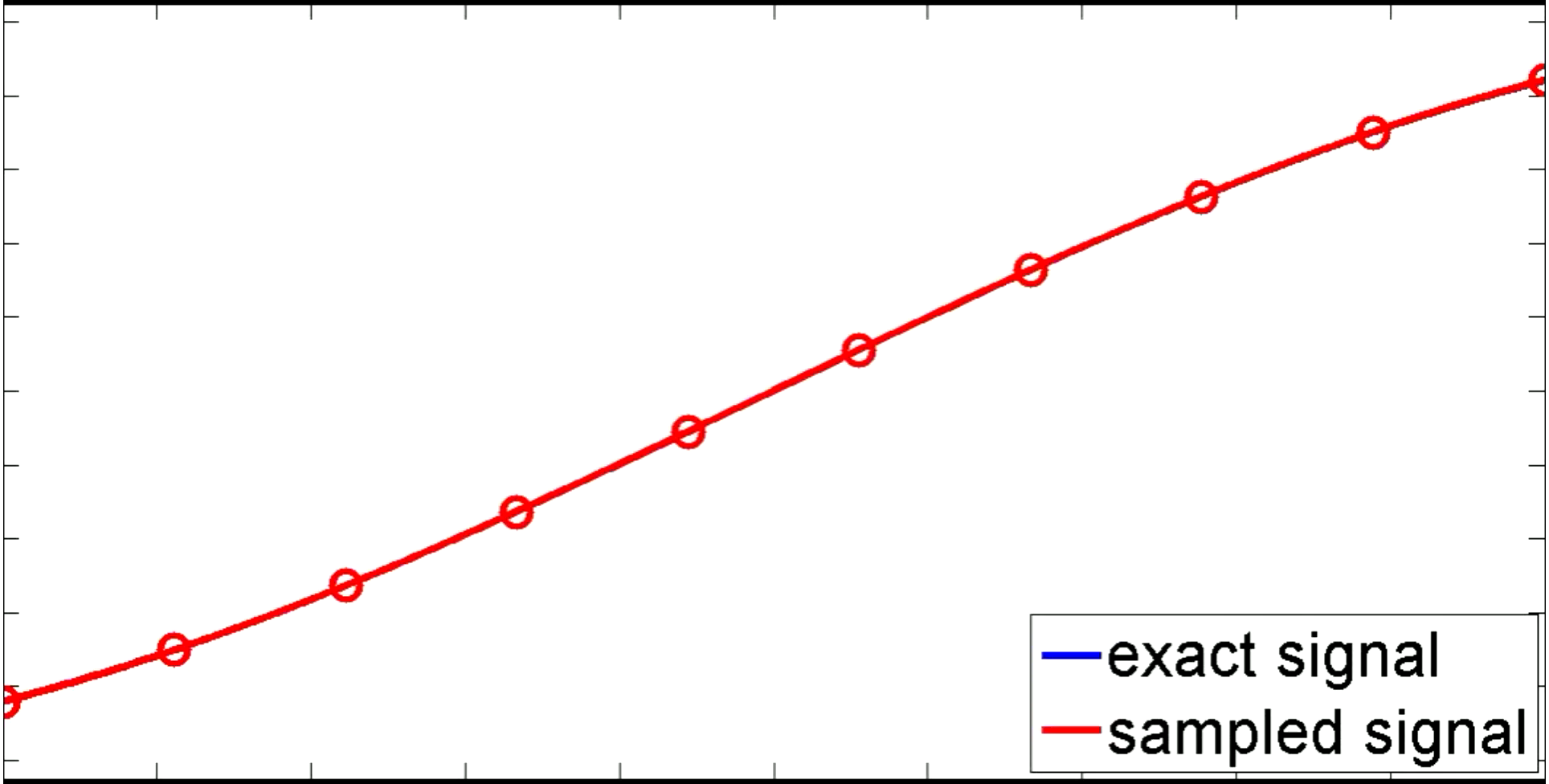
1. Sources of Instabilities
2. Local Fourier Spectral (LFS) Filters
   a) Equivalence with dissipation
   b) Previous simulations
   c) Desired properties
   d) Mechanics
   e) Simulations in coarse unstructured grids
3. Conclusion

Sources of Instabilities

1. Sharp gradients lead to Gibbs phenomenon: oscillations bring density or pressure below zero

2. Aliasing leads to incorrect physics: small scales are seen as large scales

Sources of Instabilities



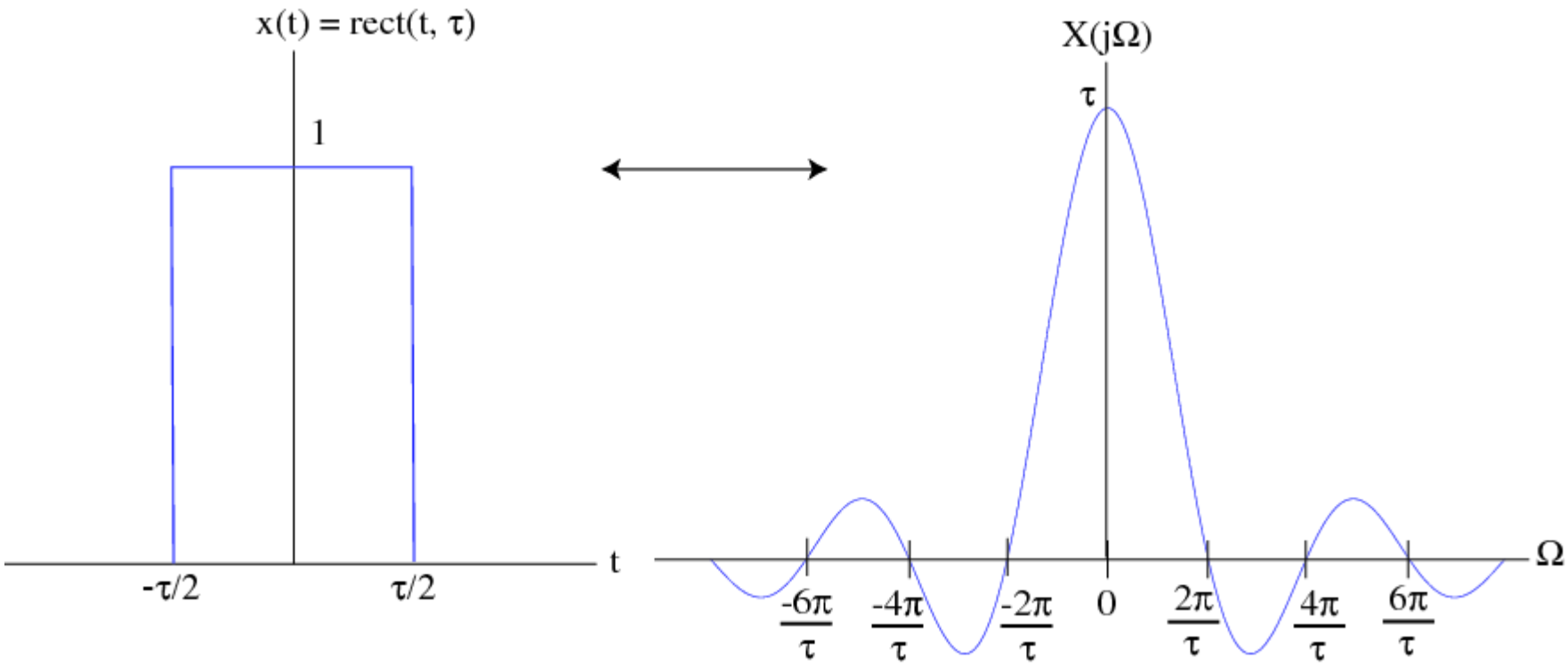
Sources of Instabilities

Similarity between sharp gradients and turbulent flows:

high range of scales

Sources of Instabilities

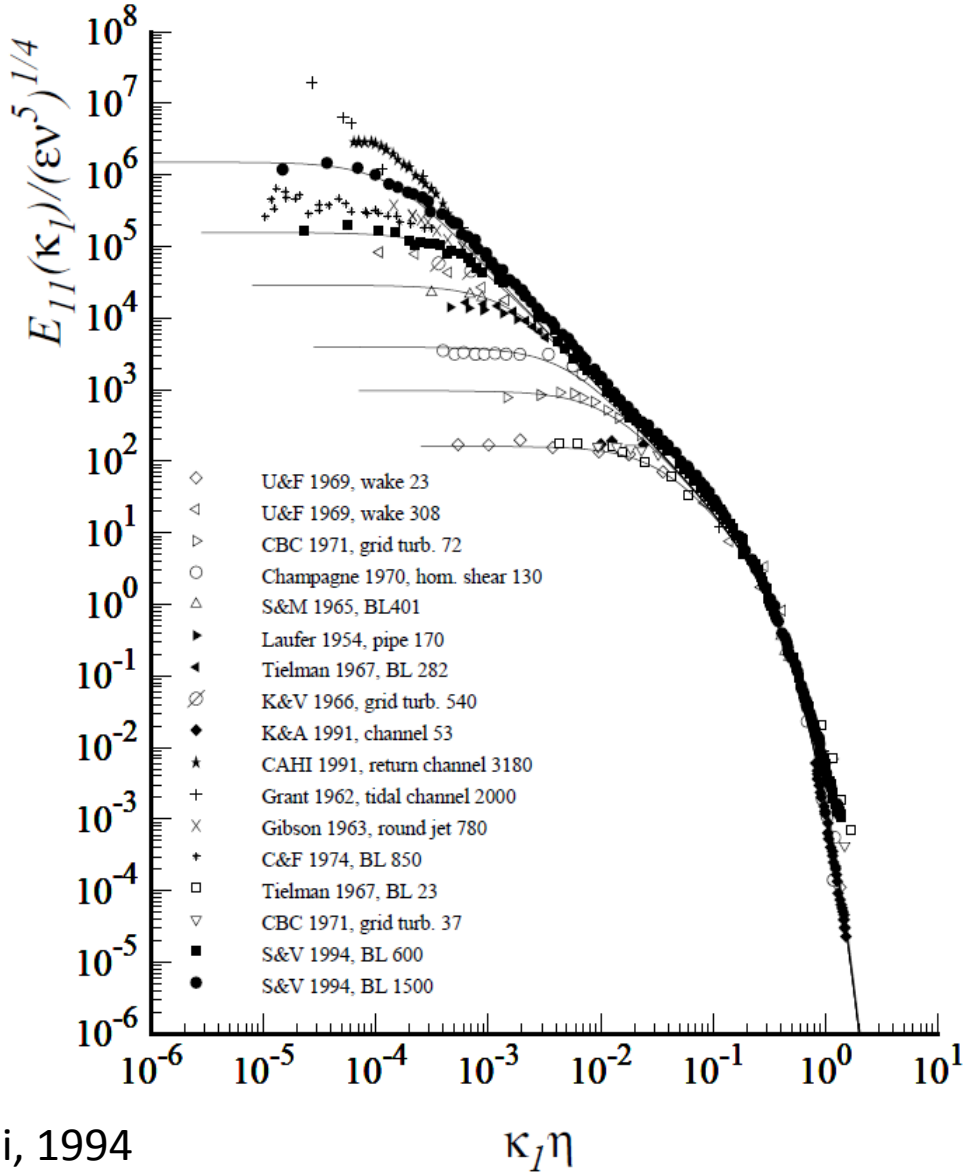
Sharp gradients



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Sources of Instabilities

Energy cascade



Saddoughi and Veeravalli, 1994

Increasing robustness

Options:

- Grid refinement
- Limiting
- Artificial dissipation
- Numerical schemes with robustness built-in
- Filtering (simplest to implement in HiFiLES)

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Stability of Flux Reconstruction

- For linear fluxes, FR can be made stable
- But for non-linear fluxes in the advection equation, aliasing becomes a problem:

$$\frac{1}{2}\frac{d}{dt}|u^{\delta}|^{2}_{\Omega^{\delta},(P,2)} \leq \sum_{n=0}^{N-1}\int_{\Omega_n}(f_n^{\delta}-f)\frac{\partial u^{\delta}}{\partial x}dx$$

$|u^{\delta}|_{\Omega^{\delta},(P,2)}$ is a broken Sobolev-type norm

Jameson et al., 2011

Local Fourier-spectral Filters

We could stabilize scheme by adding hyperviscosity terms

We'll see that hyperviscosity terms can be posed as filters

$$\frac{\partial u^\delta}{\partial t} + \frac{\partial I_P(f)}{\partial x} = \sum_{s=1}^{\infty} (-1)^{s+1} \epsilon_s \frac{\partial^{2s} u^\delta}{\partial x^{2s}}$$

Asthana et al., 2014

Local Fourier-spectral Filters

In the case of $u(x), f(u(x)) \in H^1(\Omega^\delta)$

$$\frac{1}{2}\frac{d}{dt}|u^\delta|^2_{\Omega^\delta,(P,2)} \leq$$

$$C_1(1+C_2)h \left\| \frac{\partial f}{\partial x} \right\|_{\Omega^\delta,0} \left\| \frac{\partial u}{\partial x} \right\|_{\Omega^\delta,0}$$

$$- \sum_{s=1}^{\infty} \epsilon_s \left\| \frac{\partial^s u^\delta}{\partial x^s} \right\|^2_{\Omega^\delta,0}$$

Asthana et al., 2014

Local Fourier-spectral Filters

$$\epsilon_1 > 0, \epsilon_j \geq 0 \; for \; j = 2, \ldots, P$$

$$\epsilon_1 \propto \frac{h^a (P+1)^b}{\Delta t_{CFL}}, \quad a < \frac{5}{2}, b > -1$$

For $h < h_{crit}(P)$,

$$\boxed{\frac{1}{2}\frac{d}{dt}|u^\delta|^2_{\Omega^\delta,(P,2)} \leq 0}$$

⟹ There *is* hope!

Asthana et al., 2014

Local Fourier-spectral Filters

Can pose addition of hyperviscositv as a Fourier filtering operation with kernel $G(k)$

$$G(k) = 1 - \sum_{s=1}^{\infty} \alpha_s k^{2s} \quad ; \qquad \alpha_s = \epsilon_s \Delta t$$

Can implement Fourier filtering via convolution

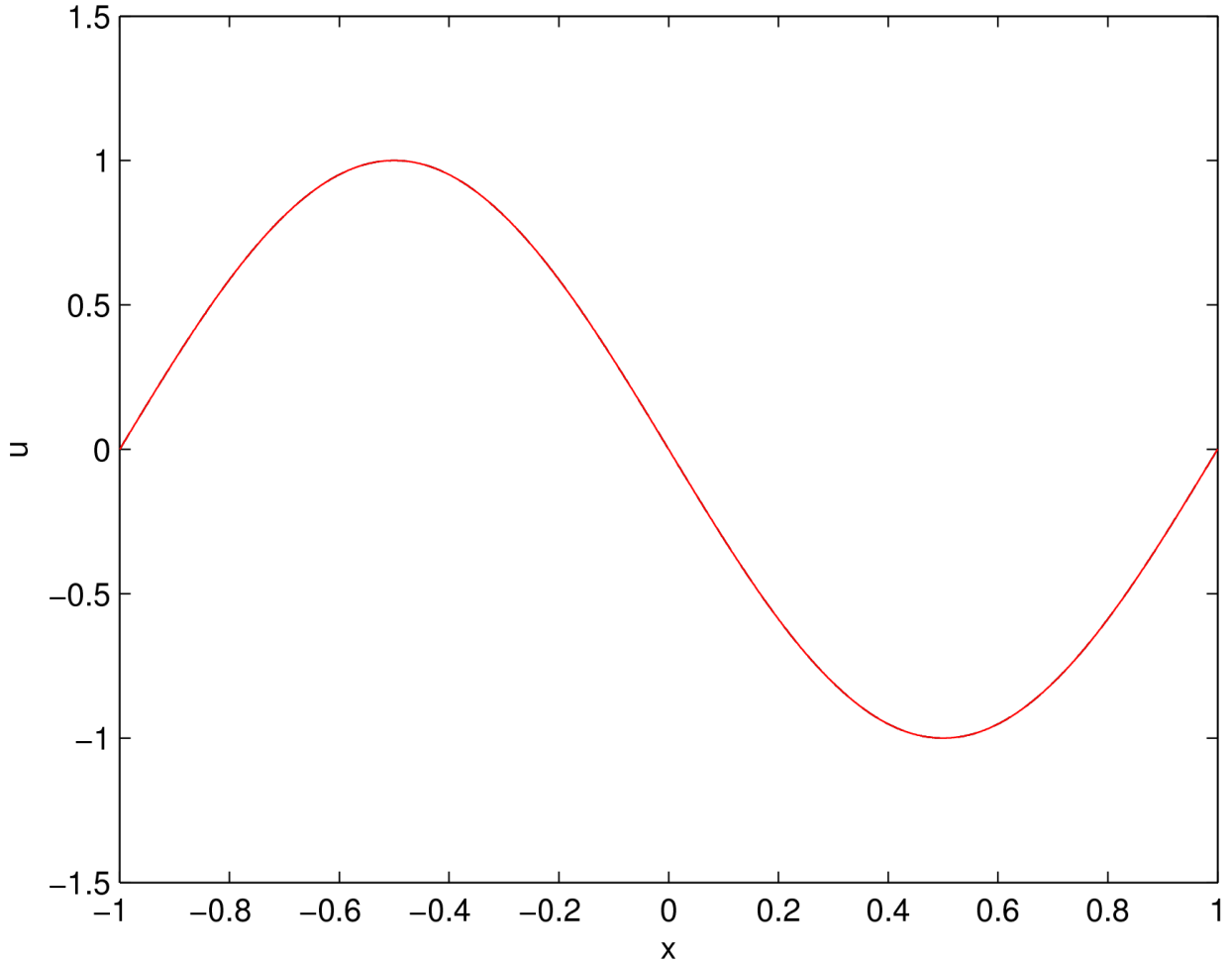
Asthana et al., 2014

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LFS Filters: tensor-product elements

Burgers; P=8;N=40; initial condition

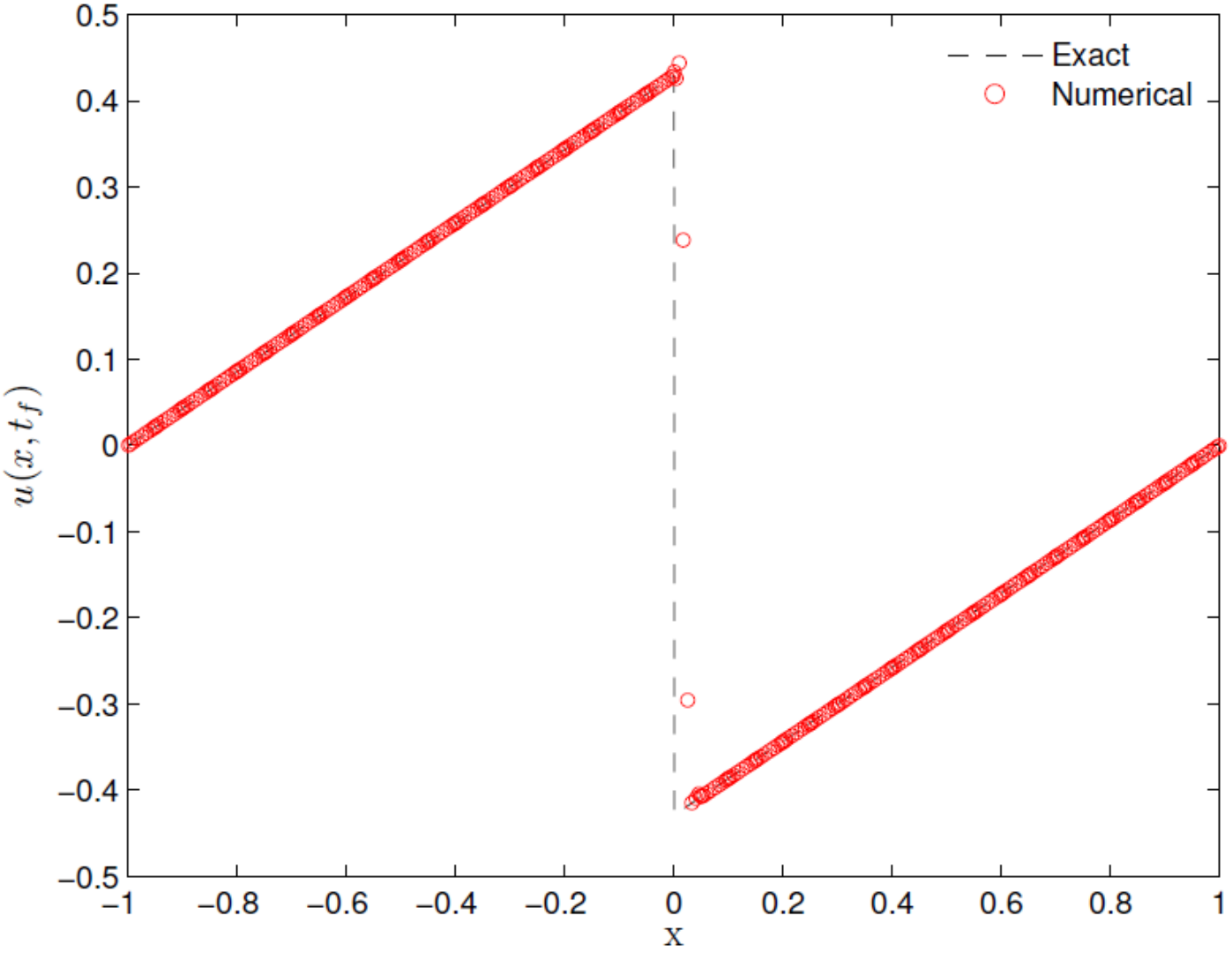


Asthana et al., 2014

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LFS Filters: tensor-product elements

Burgers; P=8;N=40; impossible without stabilization

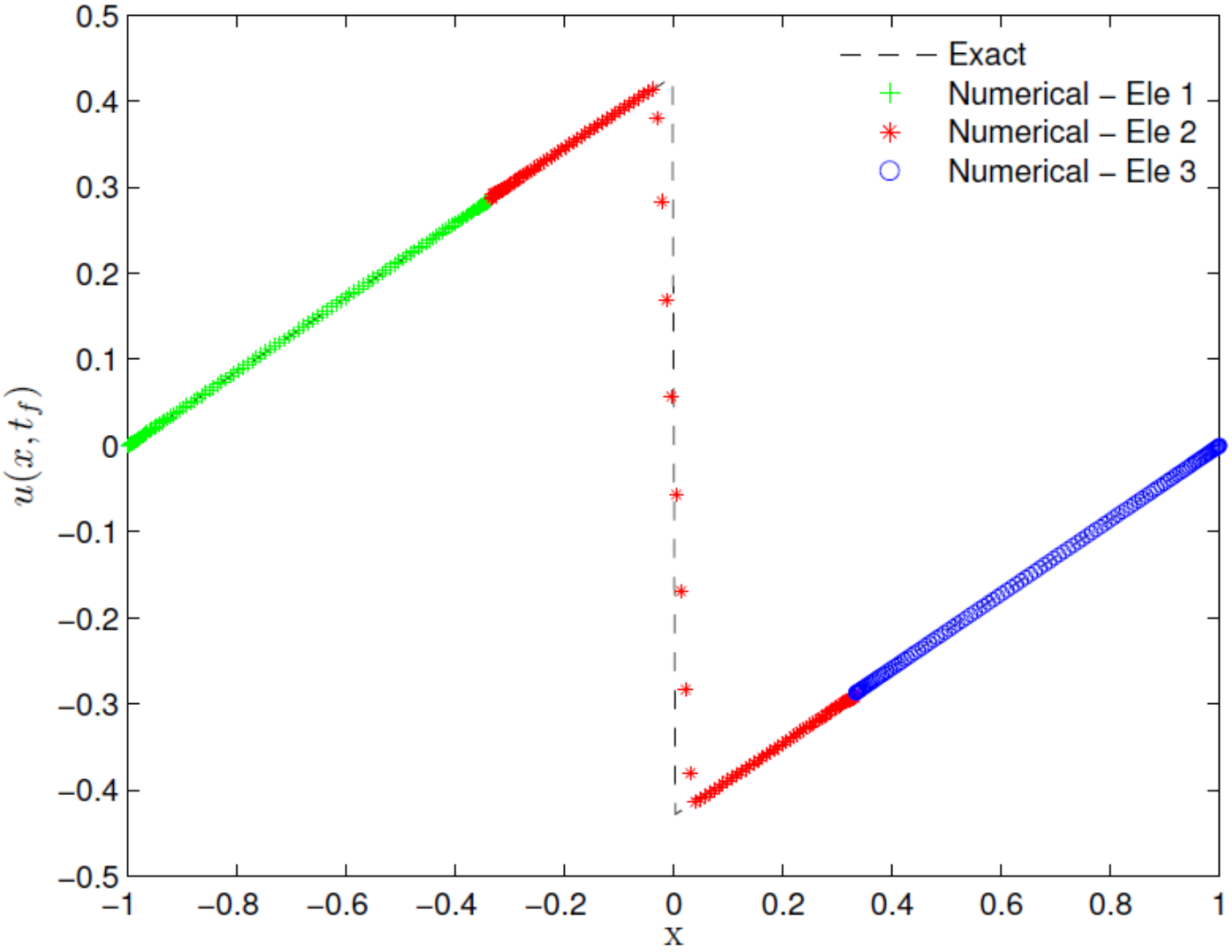


Asthana et al., 2014

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LFS Filters: tensor-product elements

Box filter; Burgers; P=119; N=3

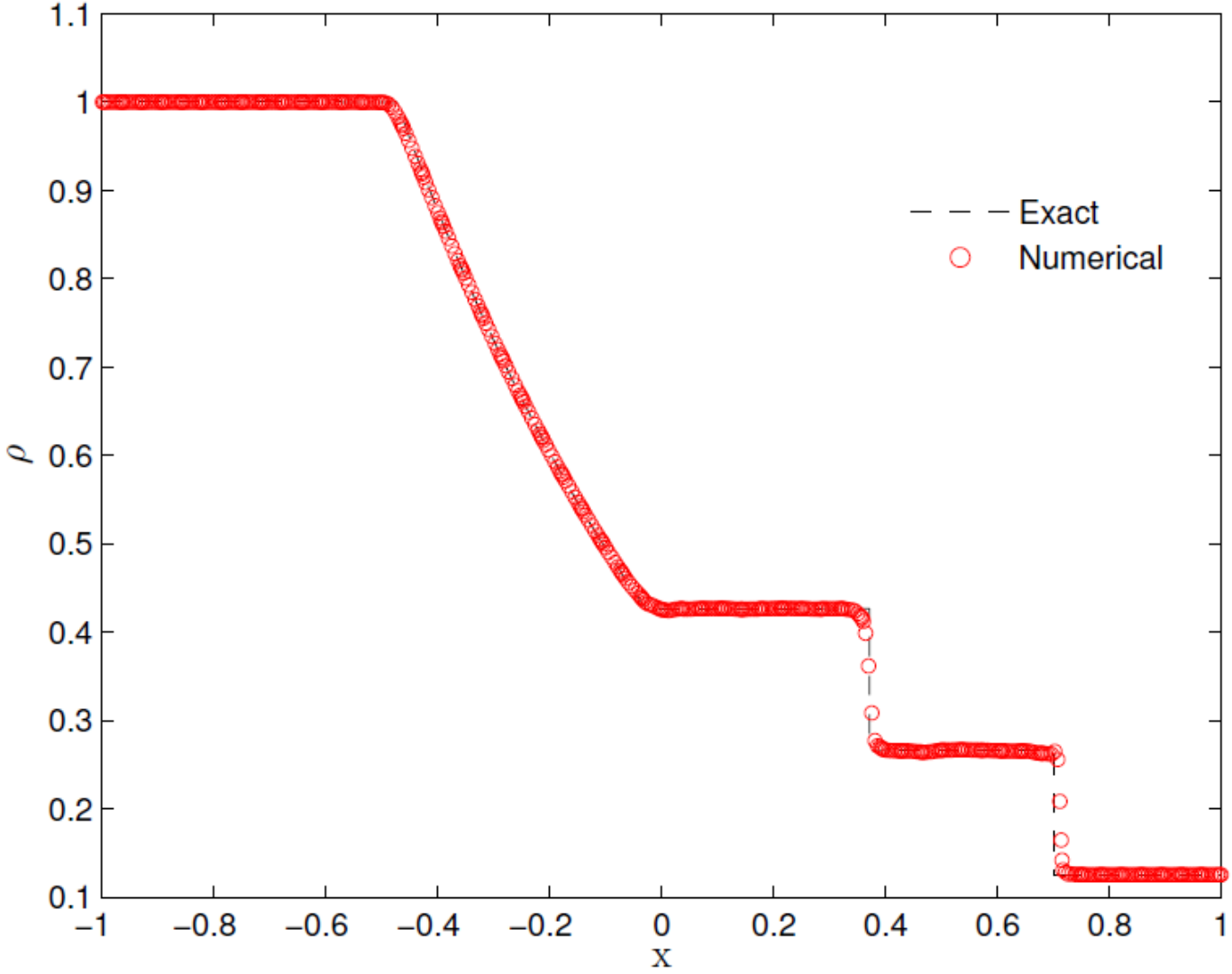


Asthana et al., 2014

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LFS Filters: tensor-product elements

Box filter; Sod's shock tube; P=8;N=56

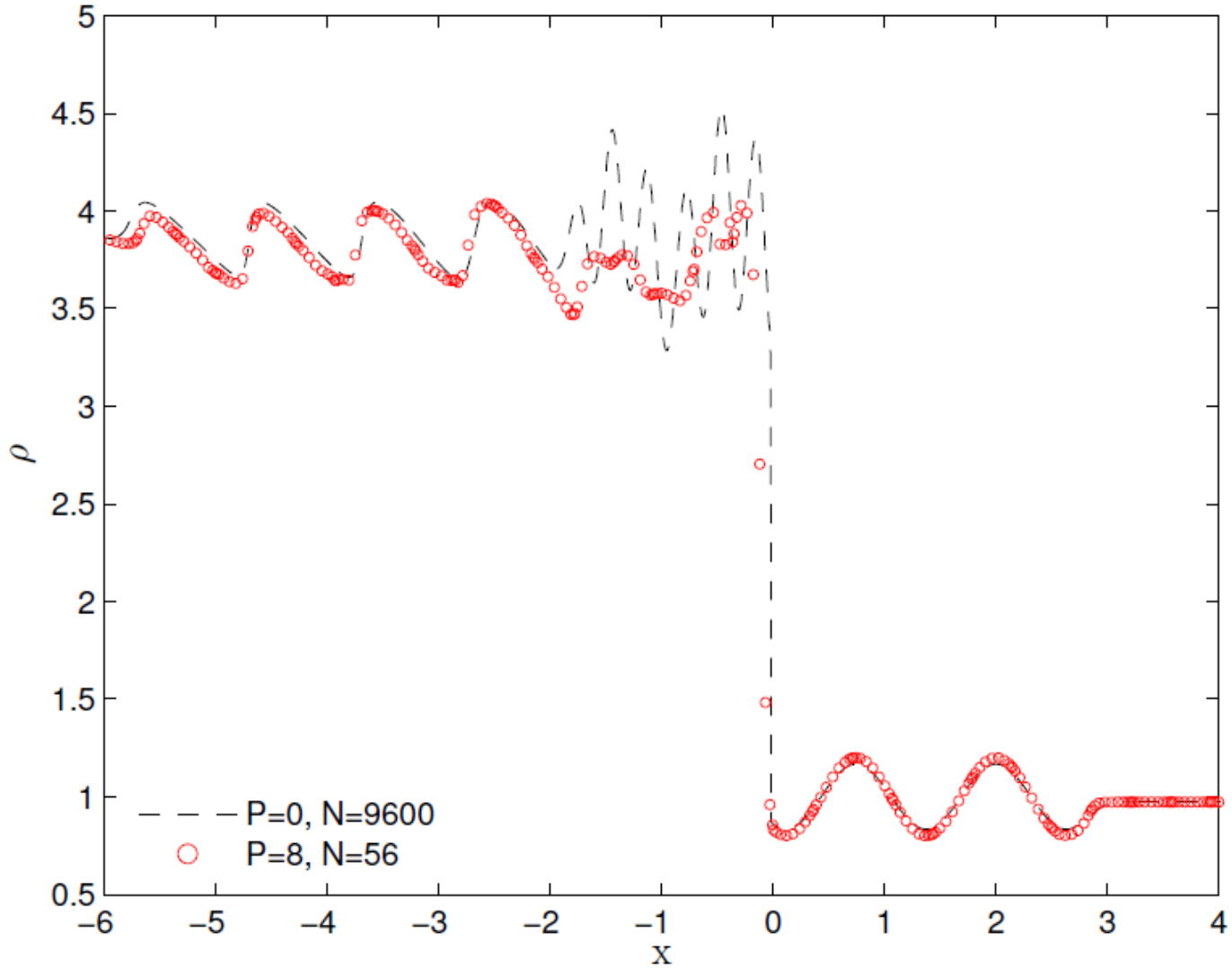


Asthana et al., 2014

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LFS Filters: tensor-product elements

Shu-Osher shock-turbulence interaction; P=8;N=56

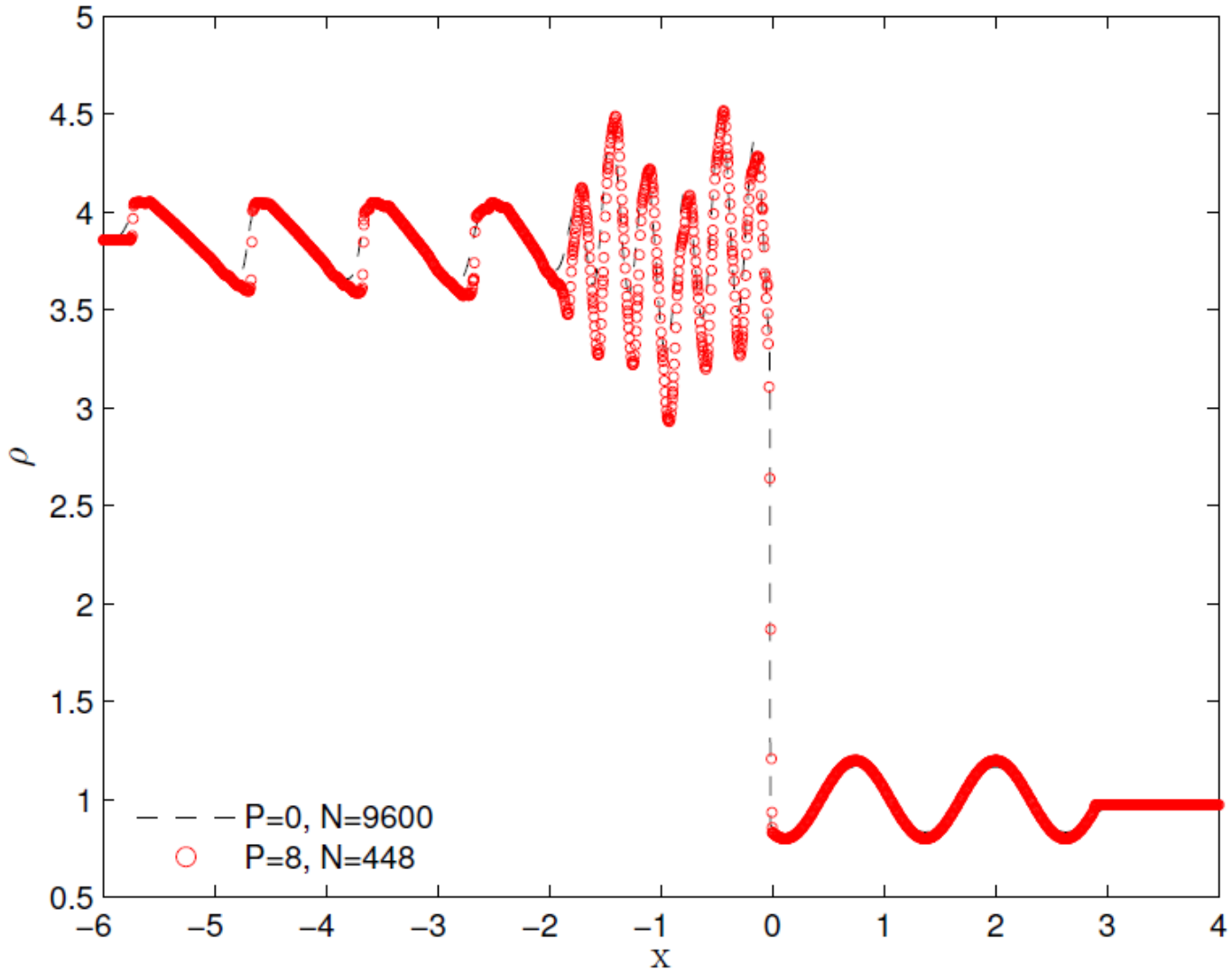


Asthana et al., 2014

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LFS Filters: tensor-product elements

Shu-Osher shock-turbulence interaction; P=8;N=448

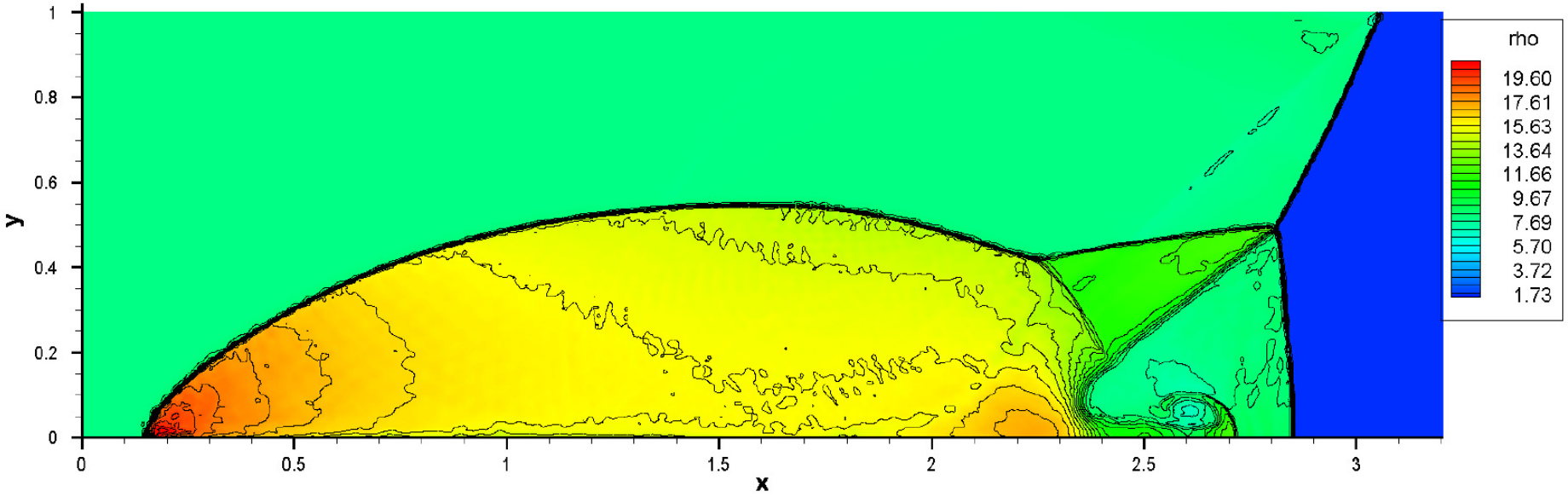


Asthana et al., 2014

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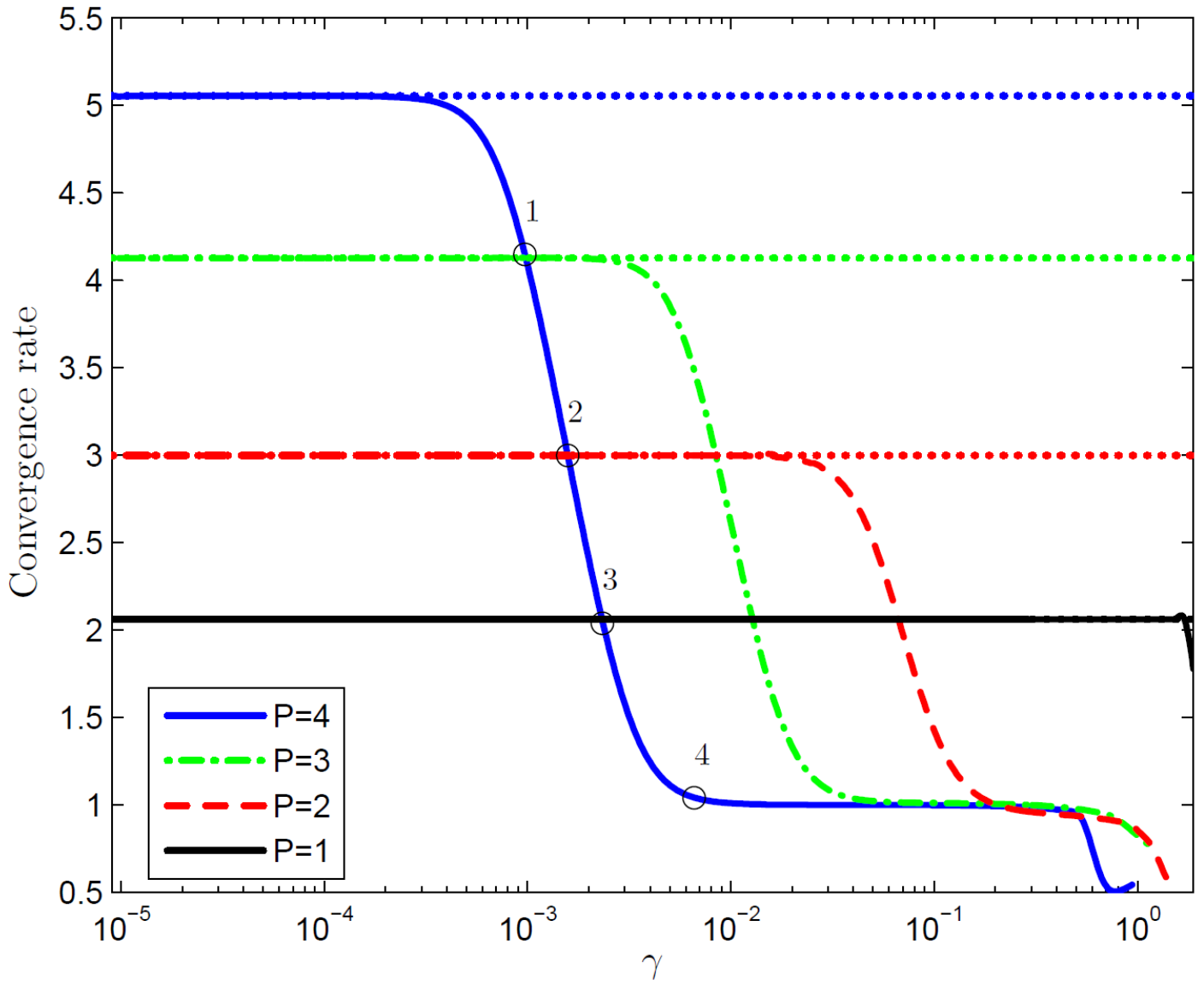
LFS Filters: tensor-product elements

Isotropic box; double Mach reflection at t = 0.1; P=8; N=56x224; density



Asthana et al., 2014

LFS Filters: order vs. accuracy



Asthana et al., 2014

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LFS Filters: order vs. accuracy

Point	$P = 1$	$P = 2$	$P = 3$	$P = 4$
1	9.57×10^{-3}	2.95×10^{-4}	6.43×10^{-6}	2.58×10^{-7}
2	9.57×10^{-3}	2.95×10^{-4}	6.44×10^{-6}	5.89×10^{-7}
3	9.57×10^{-3}	2.95×10^{-4}	6.48×10^{-6}	1.28×10^{-6}
4	9.57×10^{-3}	2.96×10^{-4}	1.02×10^{-5}	1.01×10^{-5}

Asthana et al., 2014

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LFS Filters: desired properties

Filter wish list:

1. Physical interpretation
2. Local stencil: use element-wise information only
3. Preserve boundary conditions
4. Boundary values influence internal values
5. Posed as a matrix vector multiplication
6. General to all elements
7. Preserve a linear solution

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Local Fourier-spectral Filters

$$\bar{v}_j = (G * v)(\boldsymbol{\xi}_j) \qquad v(\boldsymbol{\xi}) = \sum_{i=1}^{N_s} v_i \phi_i(\boldsymbol{\xi})$$

$$= \int_{\mathbb{R}^d} G(\boldsymbol{\xi}_j - \boldsymbol{\xi}) v(\boldsymbol{\xi}) d\boldsymbol{\xi}$$

$$= \int_{\mathbb{R}^d} G(\boldsymbol{\xi}_j - \boldsymbol{\xi}) \left(\sum_{i=1}^{N_s} v_i \phi_i(\boldsymbol{\xi}) \right) d\boldsymbol{\xi}$$

$$= \sum_{i=1}^{N_s} v_i \int_{\mathbb{R}^d} G(\boldsymbol{\xi}_j - \boldsymbol{\xi}) \phi_i(\boldsymbol{\xi}) d\boldsymbol{\xi}$$

Asthana et al., 2014

Local Fourier-spectral Filters

$$\begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_{N_s} \end{bmatrix} =$$

$$\begin{bmatrix} \int_{\mathbb{R}^d} G(\xi_1 - \xi)\phi_1(\xi)d\xi & \int_{\mathbb{R}^d} G(\xi_1 - \xi)\phi_2(\xi)d\xi & \cdots & \int_{\mathbb{R}^d} G(\xi_1 - \xi)\phi_{N_s}(\xi)d\xi \\ \int_{\mathbb{R}^d} G(\xi_2 - \xi)\phi_1(\xi)d\xi & \int_{\mathbb{R}^d} G(\xi_2 - \xi)\phi_2(\xi)d\xi & \cdots & \int_{\mathbb{R}^d} G(\xi_2 - \xi)\phi_{N_s}(\xi)d\xi \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\mathbb{R}^d} G(\xi_{N_s} - \xi)\phi_1(\xi)d\xi & \int_{\mathbb{R}^d} G(\xi_{N_s} - \xi)\phi_2(\xi)d\xi & \cdots & \int_{\mathbb{R}^d} G(\xi_{N_s} - \xi)\phi_{N_s}(\xi)d\xi \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{N_s} \end{bmatrix}$$

$$\vec{\bar{v}} = \tilde{G}\vec{v}$$

Filtering in each element becomes a matrix multiplication!

Asthana et al., 2014

Local Fourier-spectral Filters

$$\int_{\mathbb{R}^d} G(\boldsymbol{\xi}_j - \boldsymbol{\xi})\phi_i(\boldsymbol{\xi})d\boldsymbol{\xi} =$$

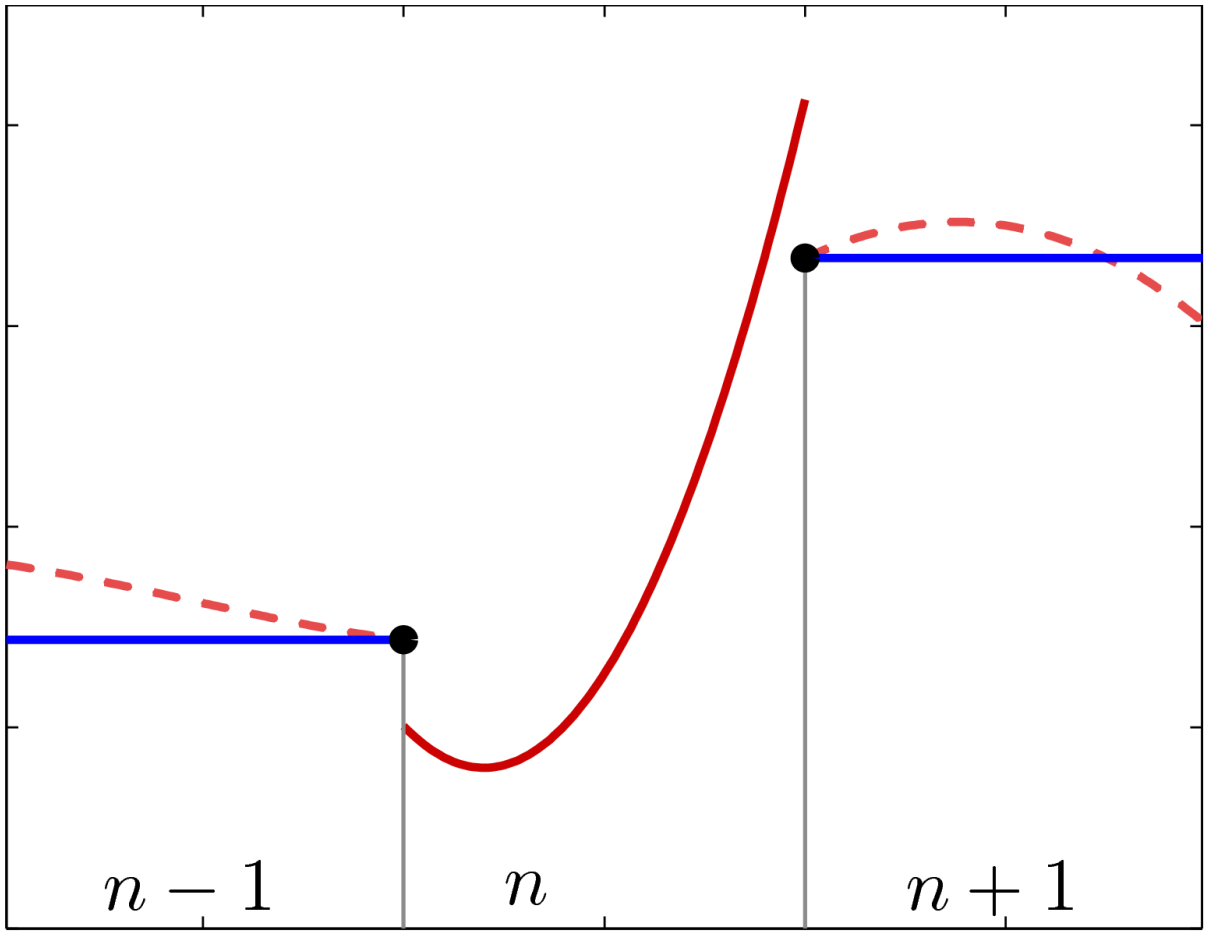
$$\int_{\Omega_n} G(\boldsymbol{\xi}_j - \boldsymbol{\xi})\phi_i(\boldsymbol{\xi})d\boldsymbol{\xi} + \int_{\mathbb{R}^d \setminus \Omega_n} G(\boldsymbol{\xi}_j - \boldsymbol{\xi})\phi_i(\boldsymbol{\xi})d\boldsymbol{\xi}$$

Basis function needs to be defined outside of Ω_n

Asthana et al., 2014

Local Fourier-spectral Filters

Possible choice in 1D:

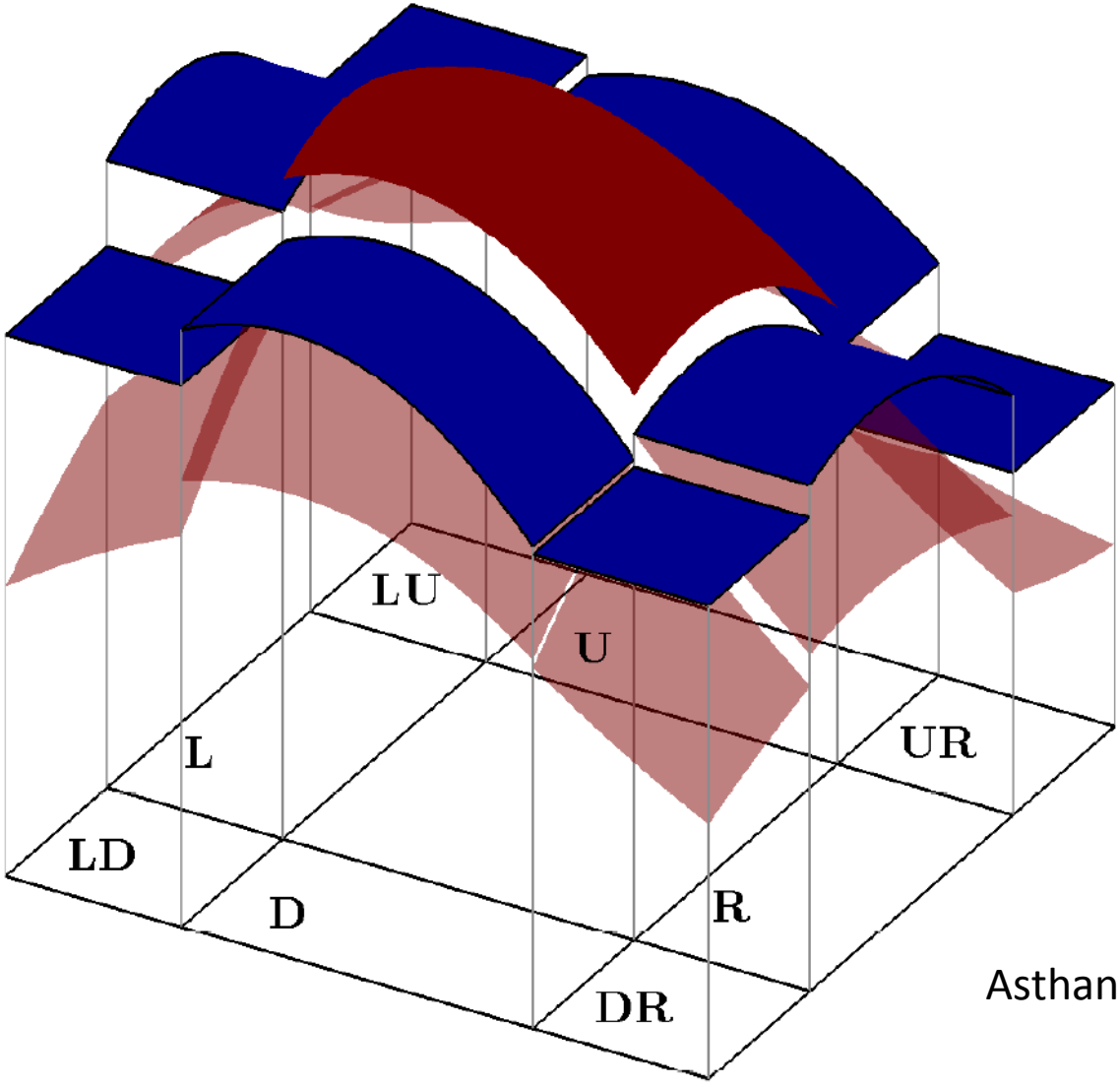


Asthana et al., 2014

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Local Fourier-spectral Filters

Possible choice in 2D:



Asthana et al., 2014

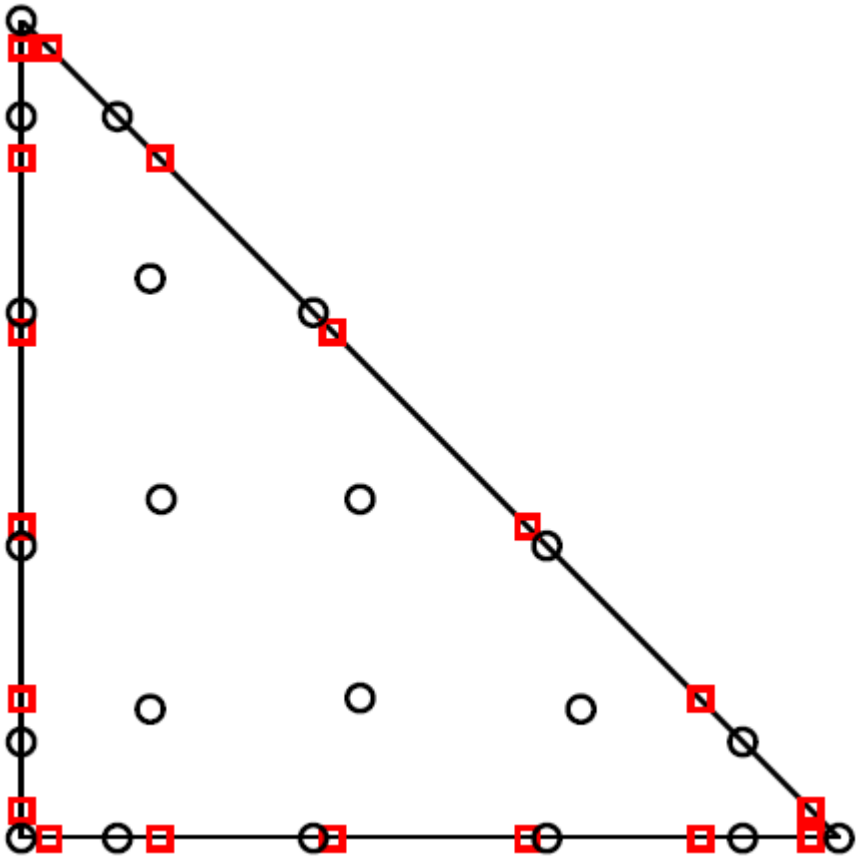
LFS Filters: general elements

Triangles, tetrahedra, pyramids, prisms

Commonality:

Internal points

Boundary points



LFS Filters: general elements

Triangles, tetrahedra, pyramids, prisms:

$$\int_{\mathbb{R}^d} G(\boldsymbol{\xi}_j - \boldsymbol{\xi})\phi_i(\boldsymbol{\xi})d\boldsymbol{\xi} =$$

$$\int_{\Omega_n} G(\boldsymbol{\xi}_j - \boldsymbol{\xi})\phi_i(\boldsymbol{\xi})d\boldsymbol{\xi} + \int_{\mathbb{R}^d \setminus \Omega_n} G(\boldsymbol{\xi}_j - \boldsymbol{\xi})\phi_i(\boldsymbol{\xi})d\boldsymbol{\xi}$$

Internal Component Boundary Component

$$\vec{v} = \alpha \mathcal{T} \vec{v} + (1 - \alpha)\mathcal{B}\vec{v^*}$$

LFS Filters: general elements

Internal Component Boundary Component

$$\vec{v} = \alpha \mathcal{T} \vec{v} + (1 - \alpha) \mathcal{B} \vec{v^*}$$

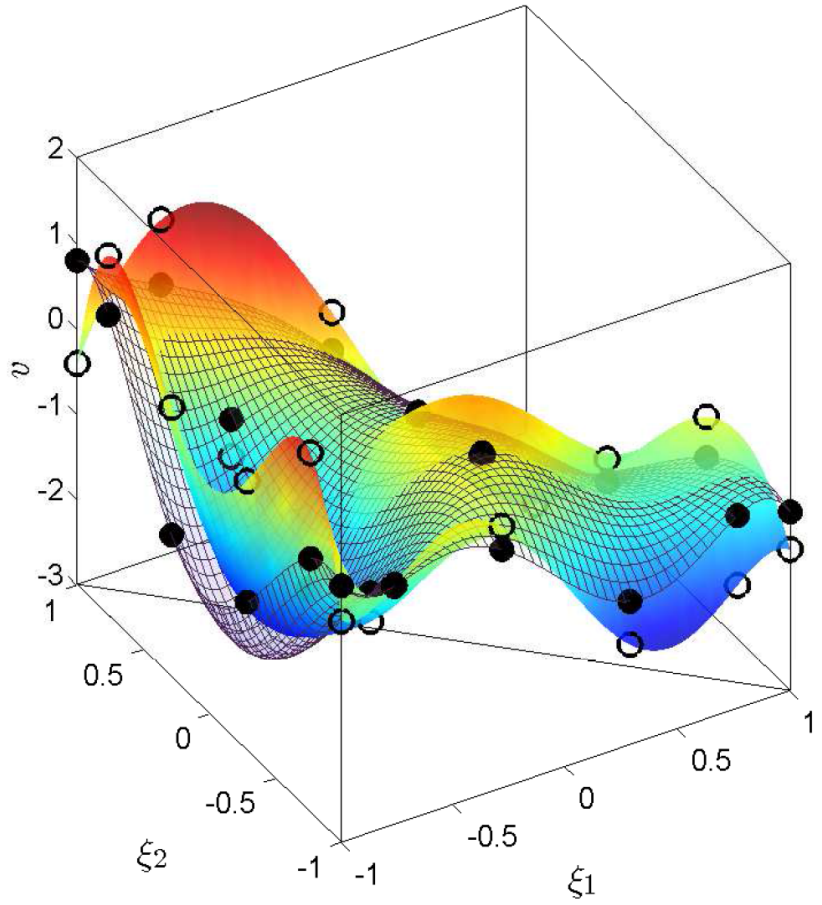
$$\mathcal{T}_{ij} = \int_{\Omega_n} G(\boldsymbol{\xi}_i - \boldsymbol{\xi}) \phi_j(\boldsymbol{\xi}) d\boldsymbol{\xi}$$

Integral can be evaluated to arbitrary accuracy

LFS Filters: general elements

Internal Component Boundary Component

$$\vec{\vec{v}} = \alpha \mathcal{T} \vec{v} + (1 - \alpha) \mathcal{B} \vec{v^*}$$



Using a sharp-spectral filter

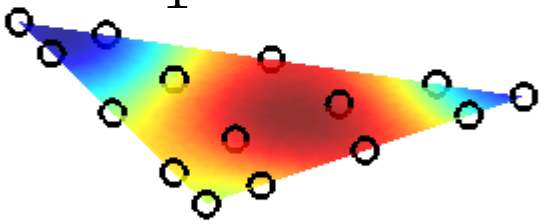
LFS Filters: general elements

Internal Component Boundary Component

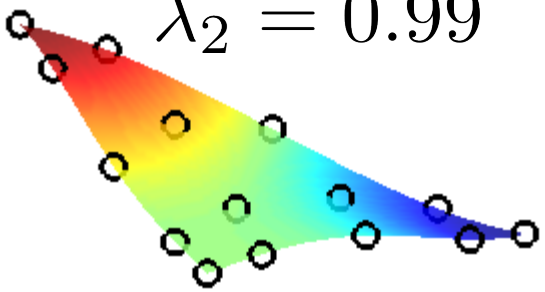
$$\vec{v} = \alpha \mathcal{T} \vec{v} + (1 - \alpha) \mathcal{B} \vec{v^*}$$

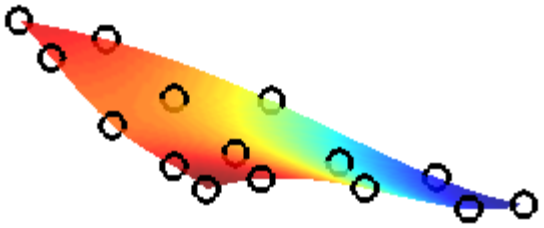
Let us look at some of the modes (eigenvectors of $\mathcal{T}$) being filtered (corresponding eigenvalues)

$\lambda_1 = 1.0$

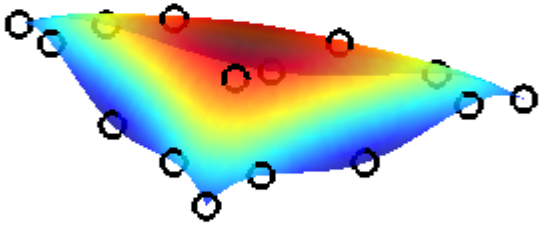


$\lambda_2 = 0.99$





$\lambda_3 = 0.99$



$\lambda_2 = 0.64$

LFS Filters: general elements

Internal Component **Boundary Component**

$$\vec{\vec{v}} = \alpha \mathcal{T} \vec{v} + (1 - \alpha) \mathcal{B} \vec{v^*}$$

Requires more care.

In order to preserve linear solution, assume

$$\vec{v^*} = \underline{\boldsymbol{\xi}^{*\mathrm{T}}} \vec{a} + \vec{b^*}$$

LFS Filters: general elements

Internal Component **Boundary Component**

$$\vec{\vec{v}} = \alpha \mathcal{T} \vec{v} + (1 - \alpha) \mathcal{B} \vec{v^*}$$

In order to preserve linear solution, assume

$$\vec{v^*} = \underline{\boldsymbol{\xi}^{*\mathrm{T}}} \vec{a} + \vec{b^*}$$

When $\alpha = 0$, we would like to obtain

$$\vec{\vec{v}} = \mathcal{B} \vec{v^*} = \underline{\boldsymbol{\xi}^{\mathrm{T}}} \vec{a} + \vec{b}$$

LFS Filters: general elements

Internal Component **Boundary Component**

$$\vec{\vec{v}} = \alpha \mathcal{T} \vec{v} + (1 - \alpha) \mathcal{B} \vec{v^*}$$

Preservation of linear solution requires

$$\sum_{j=1}^{N_f} \mathcal{B}_{ij} \xi^*_{jd} = \xi_{id} \qquad \sum_{j=1}^{N_f} \mathcal{B}_{ij} = 1$$

LFS Filters: general elements

Internal Component **Boundary Component**

$$\vec{\vec{v}} = \alpha \mathcal{T} \vec{v} + (1 - \alpha) \mathcal{B} \vec{v^*}$$

Enforce one condition per row i: $\sum_{j=1}^{N_f} \mathcal{B}_{ij} = 1$

$$\mathcal{B}_{ij} = \frac{h_i\left(||\boldsymbol{\xi}_i - \boldsymbol{\xi}_j^*||\right)^{-1}}{\sum_{k=1}^{N_f} h_i\left(||\boldsymbol{\xi}_i - \boldsymbol{\xi}_k^*||\right)^{-1}}$$

LFS Filters: general elements

Internal Component **Boundary Component**

$$\vec{v} = \alpha \mathcal{T} \vec{v} + (1 - \alpha) \mathcal{B} \vec{v^*}$$

Enforce one condition per row i : $\sum_{j=1}^{N_f} \mathcal{B}_{ij} \xi^*_{jd} = \xi_{id}$

Introduce unknowns a_{id}

$$h_i(x) = \sum_{d=1}^{N_d} a_{id} x^d$$

LFS Filters: general elements

Internal Component **Boundary Component**

$$\vec{\vec{v}} = \alpha \mathcal{T} \vec{v} + (1 - \alpha) \mathcal{B} \vec{v^*}$$

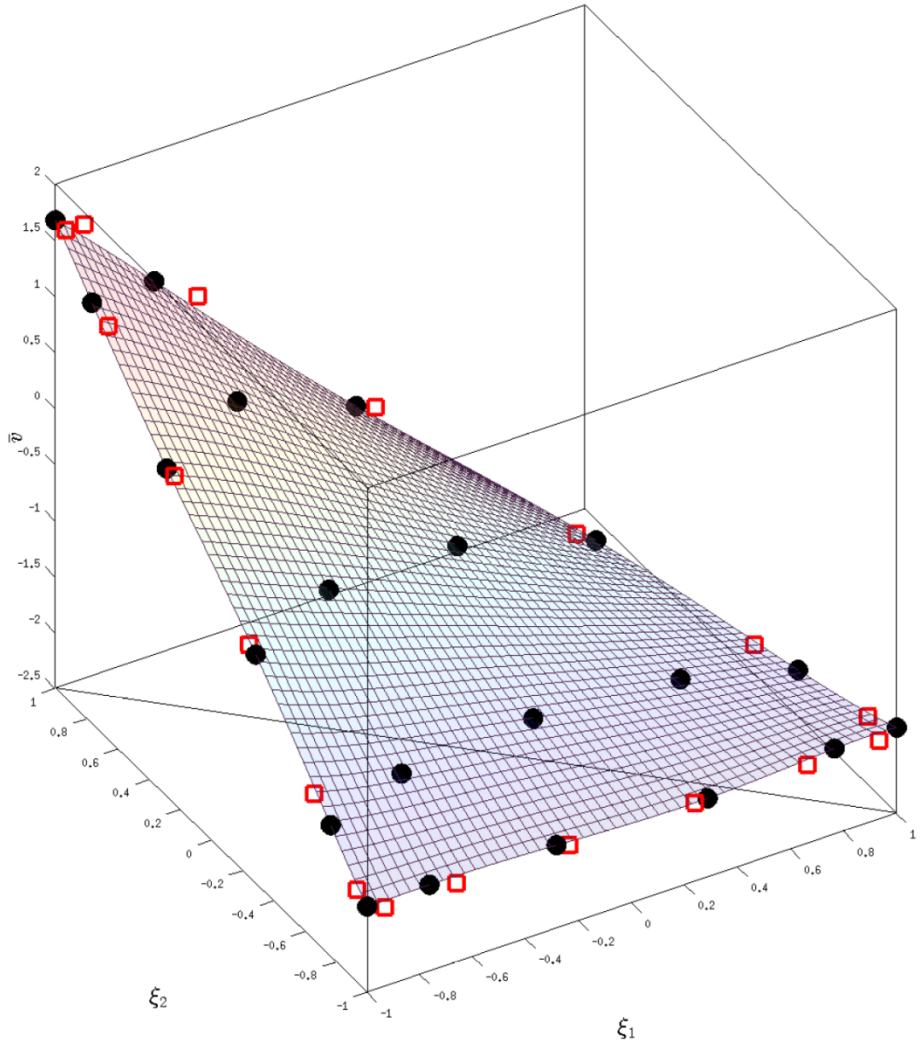
Enforce one condition per row: $\sum_{j=1}^{N_f} \mathcal{B}_{ij} \xi^*_{jd} = \xi_{id}$

Minimize:

$$J(a_{i1}, \ldots, a_{1N_d}) = \sum_{d=1}^{N_d} \left(\sum_{j=1}^{N_f} \mathcal{B}_{ij} \xi^*_{jd} - \xi_{id} \right)^2$$

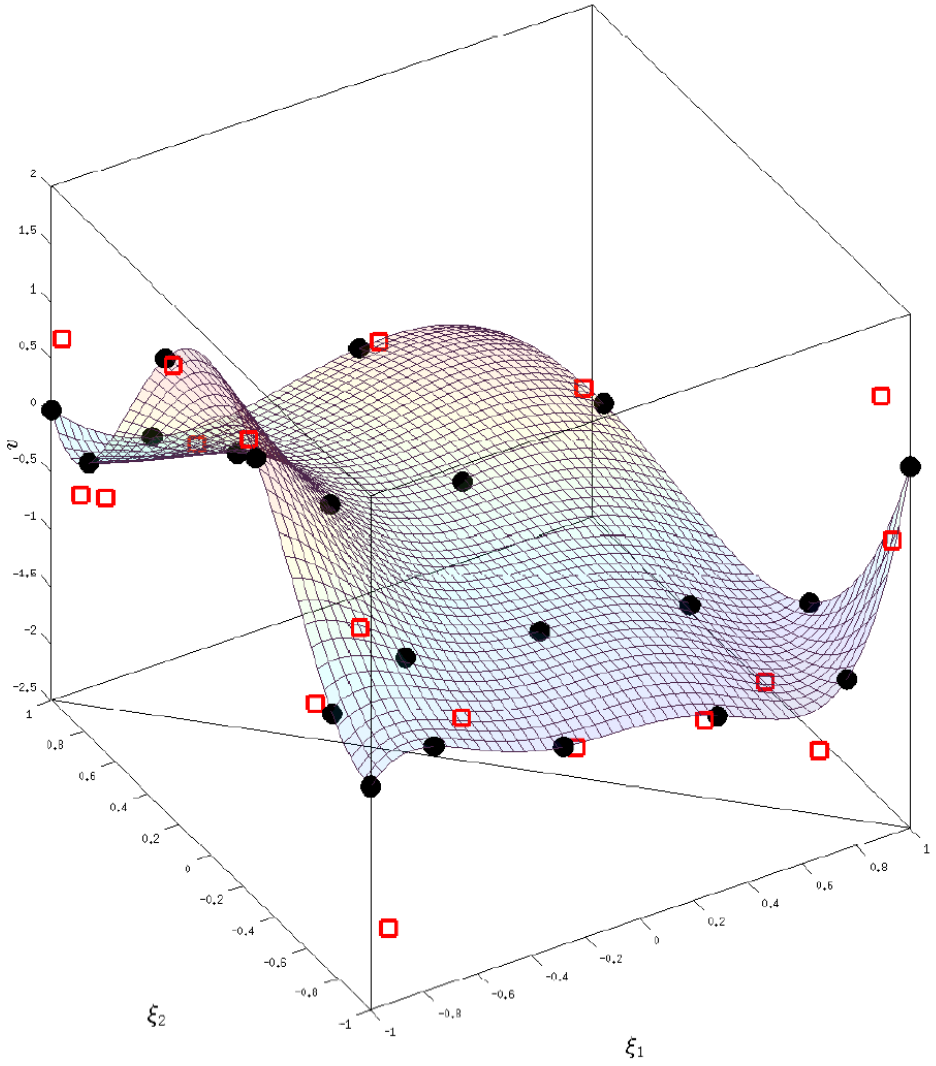
LFS Filters: general elements

Does the boundary component preserve a linear solution?



LFS Filters: general elements

Does the boundary component respect boundary influences?

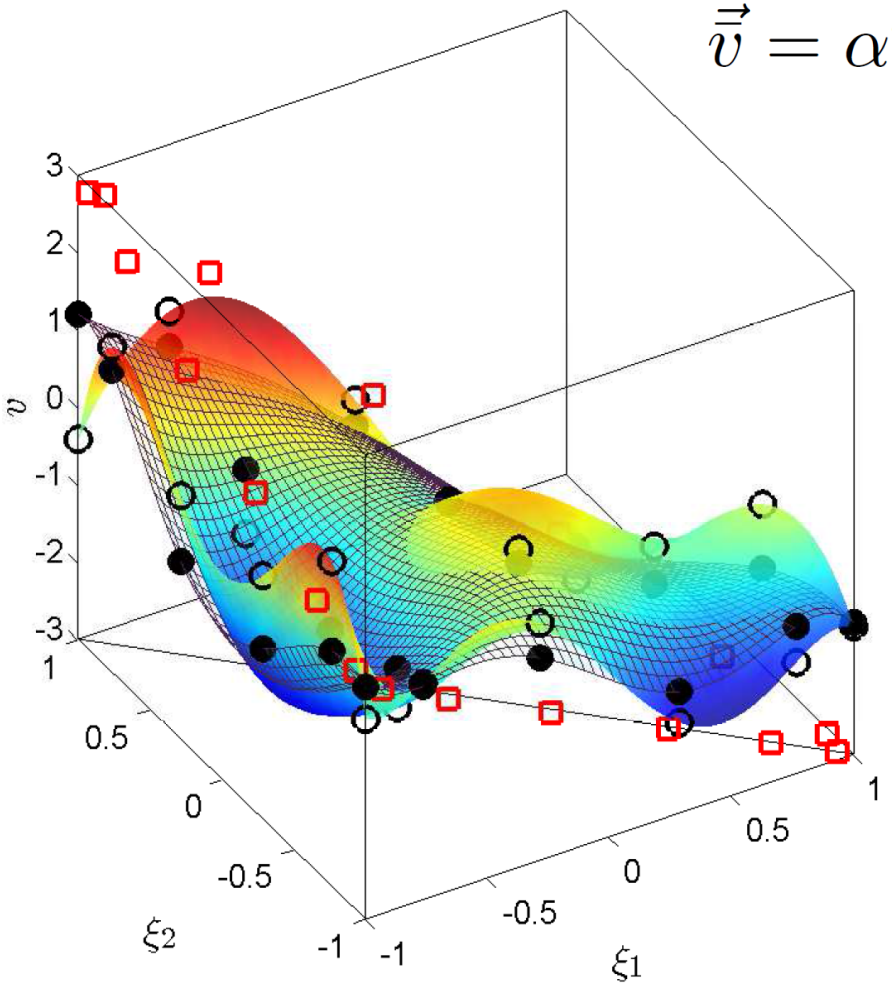


LFS Filters: general elements

What is their combined effect?

$$\vec{v} = \alpha \mathcal{T} \vec{v} + (1 - \alpha) \mathcal{B} \vec{v^*}$$

$$\alpha = 0.8$$



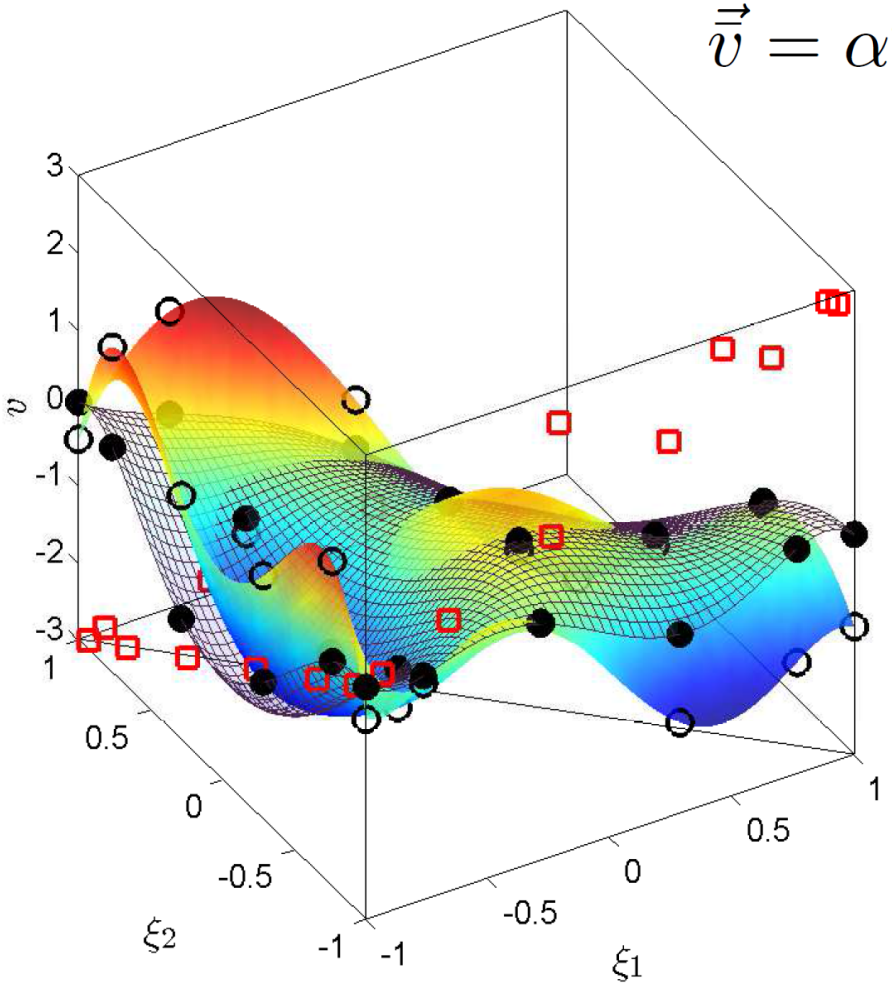
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LFS Filters: general elements

What is their combined effect?

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$$\alpha = 0.8$$



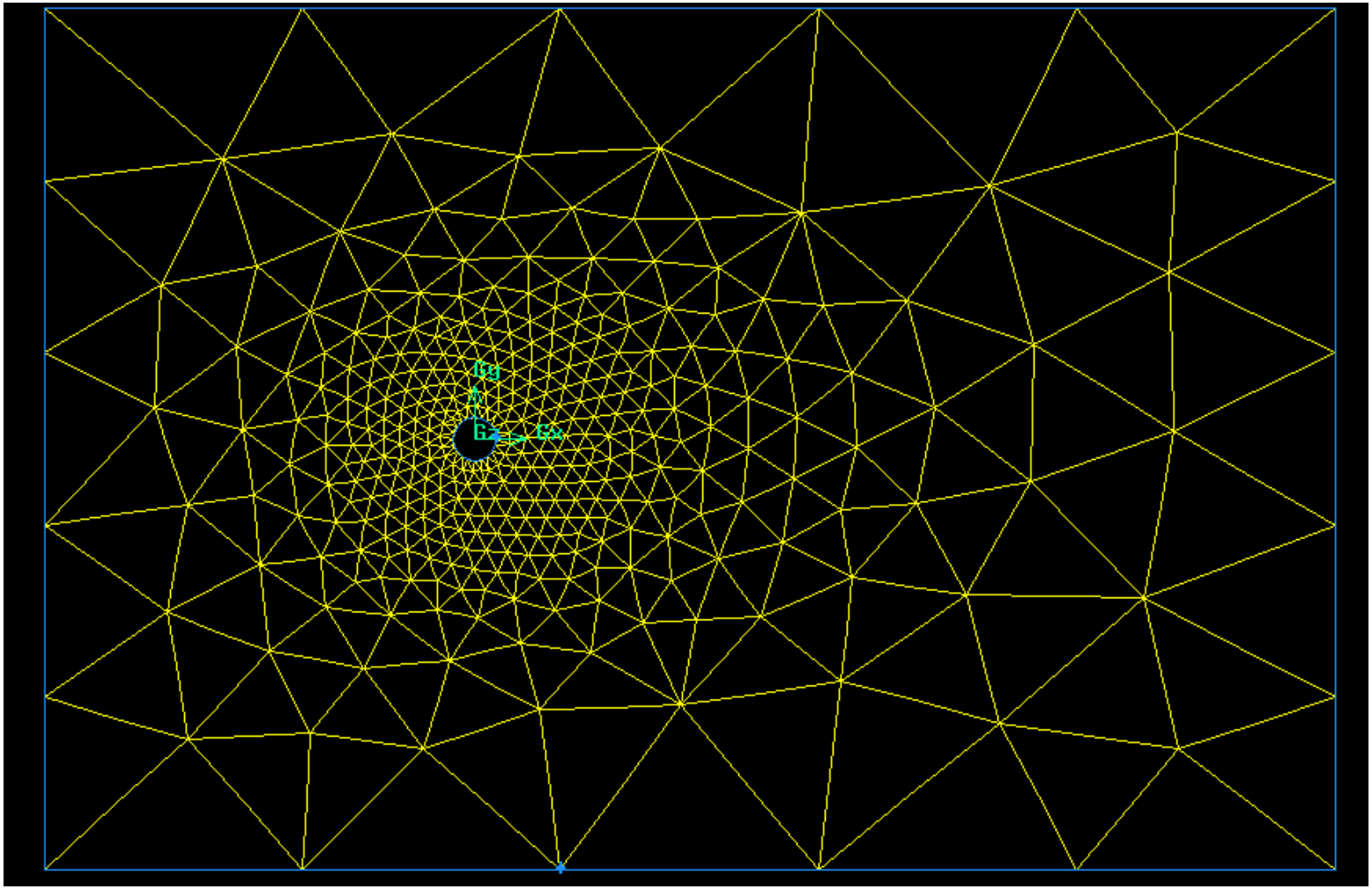
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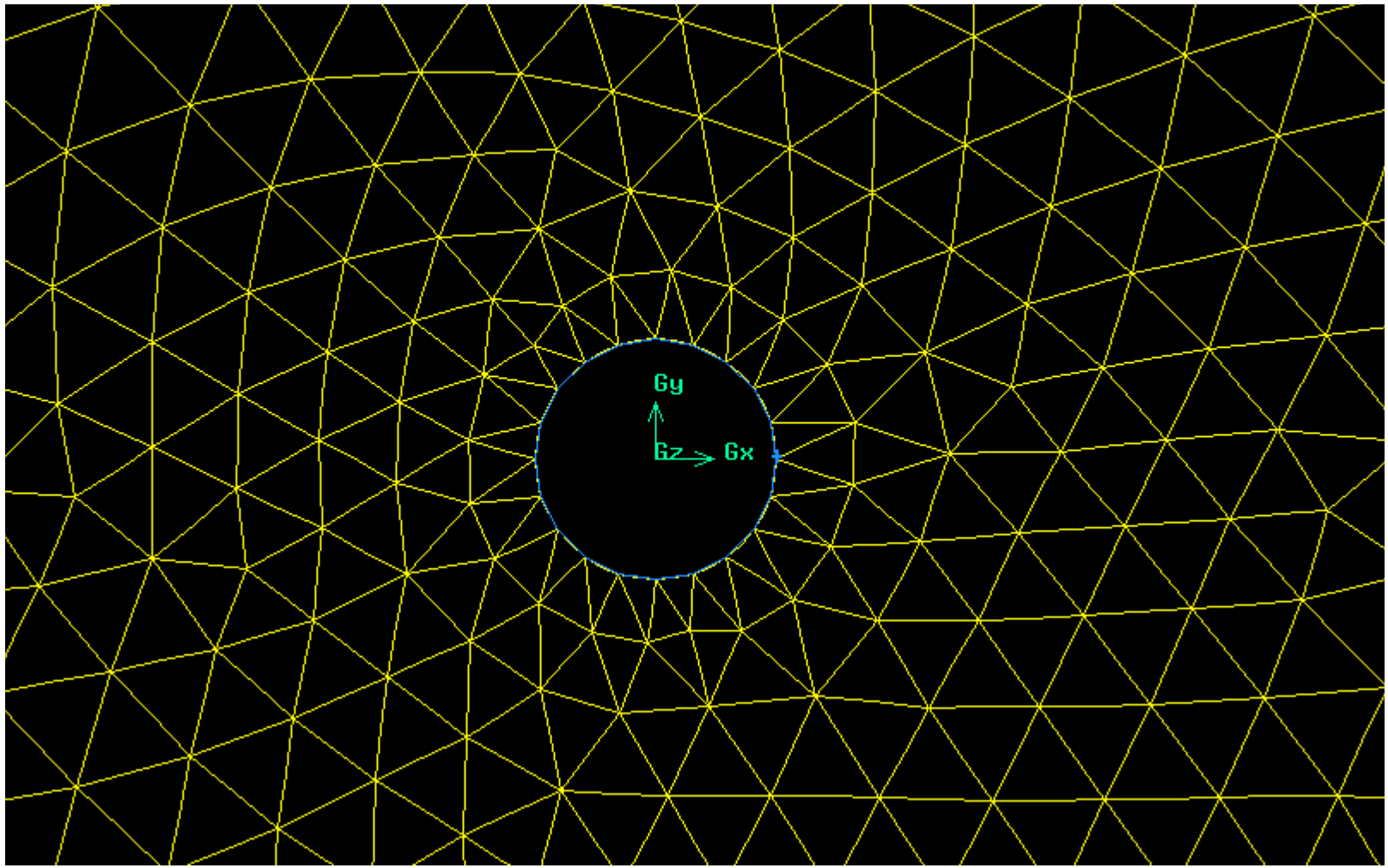
Robustness in 2D: Re = 1e6

Low quality grid for such high Re number

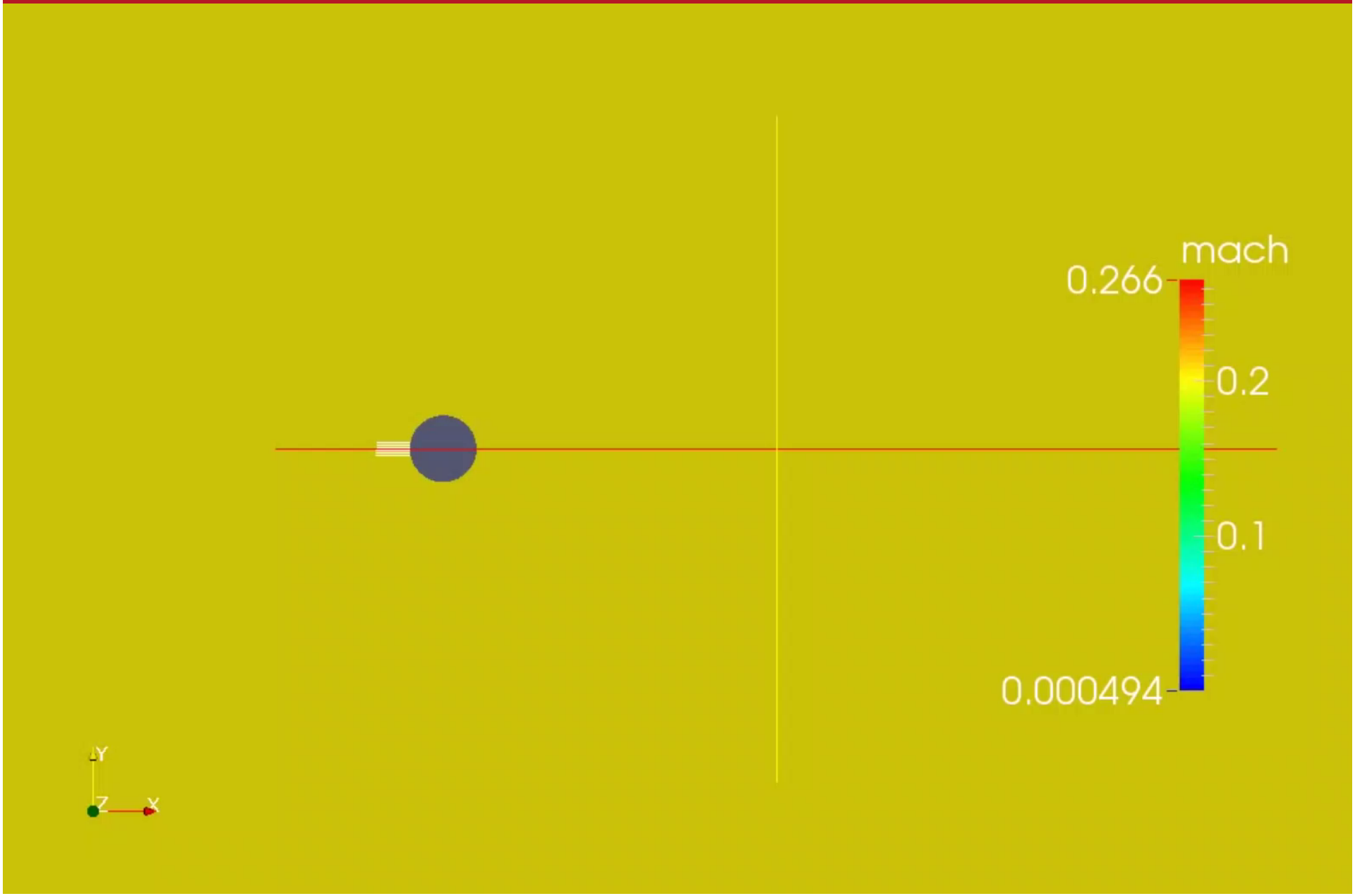


Robustness in 2D: Re = 1e6

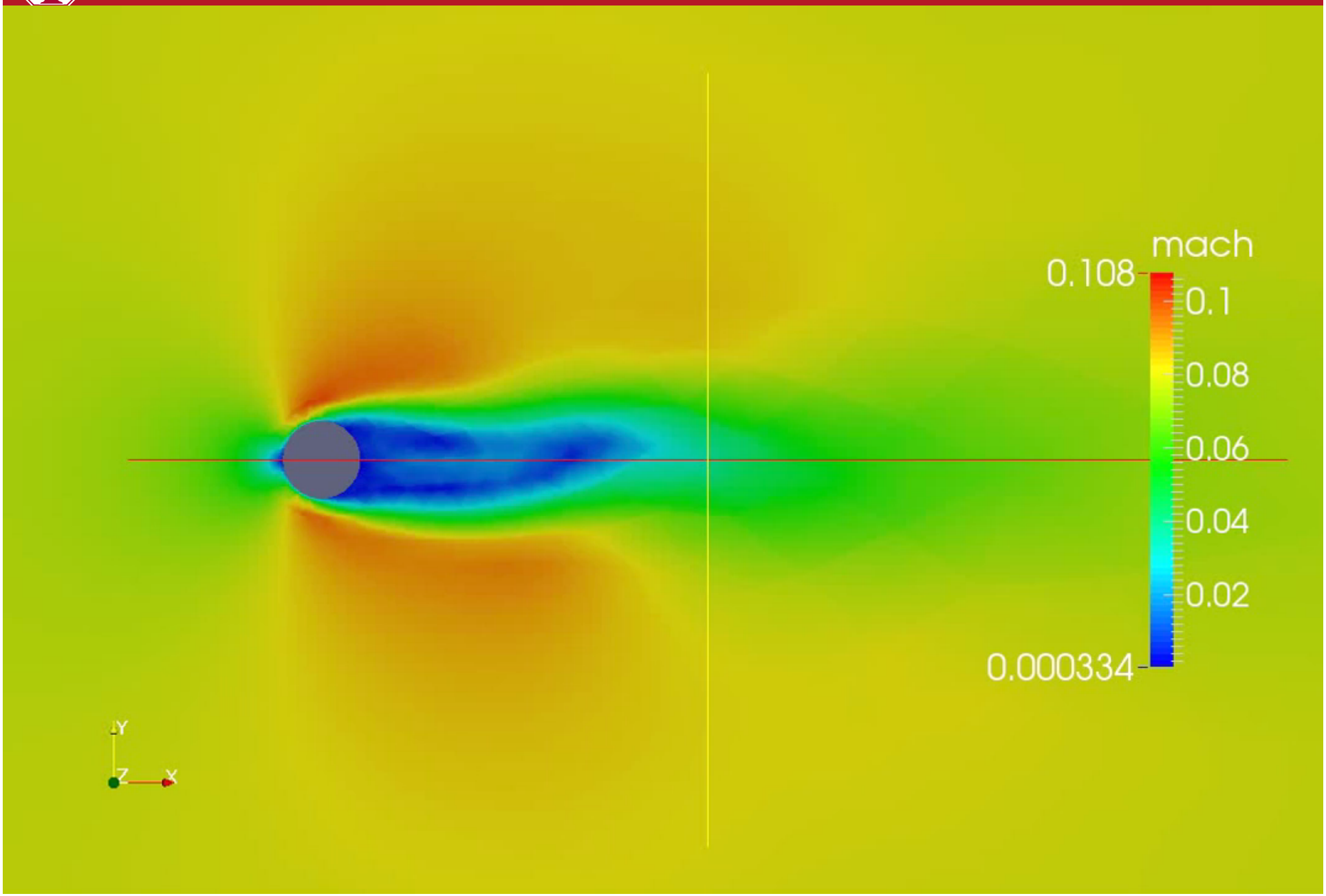
Low quality grid for such high Re number: up close



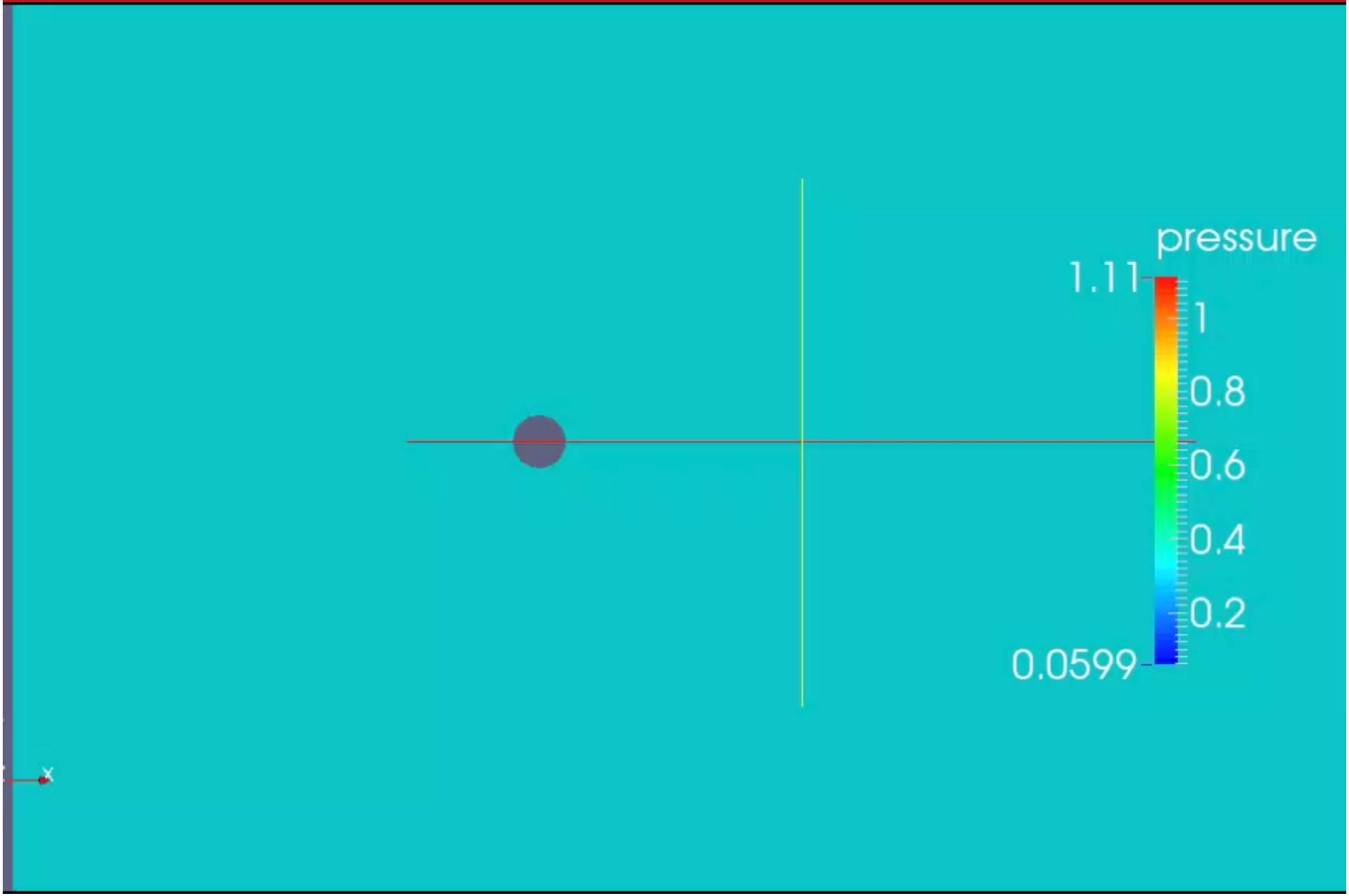
Robustness in 2D: Ma = 0.2, Re = 1e6 , P=4



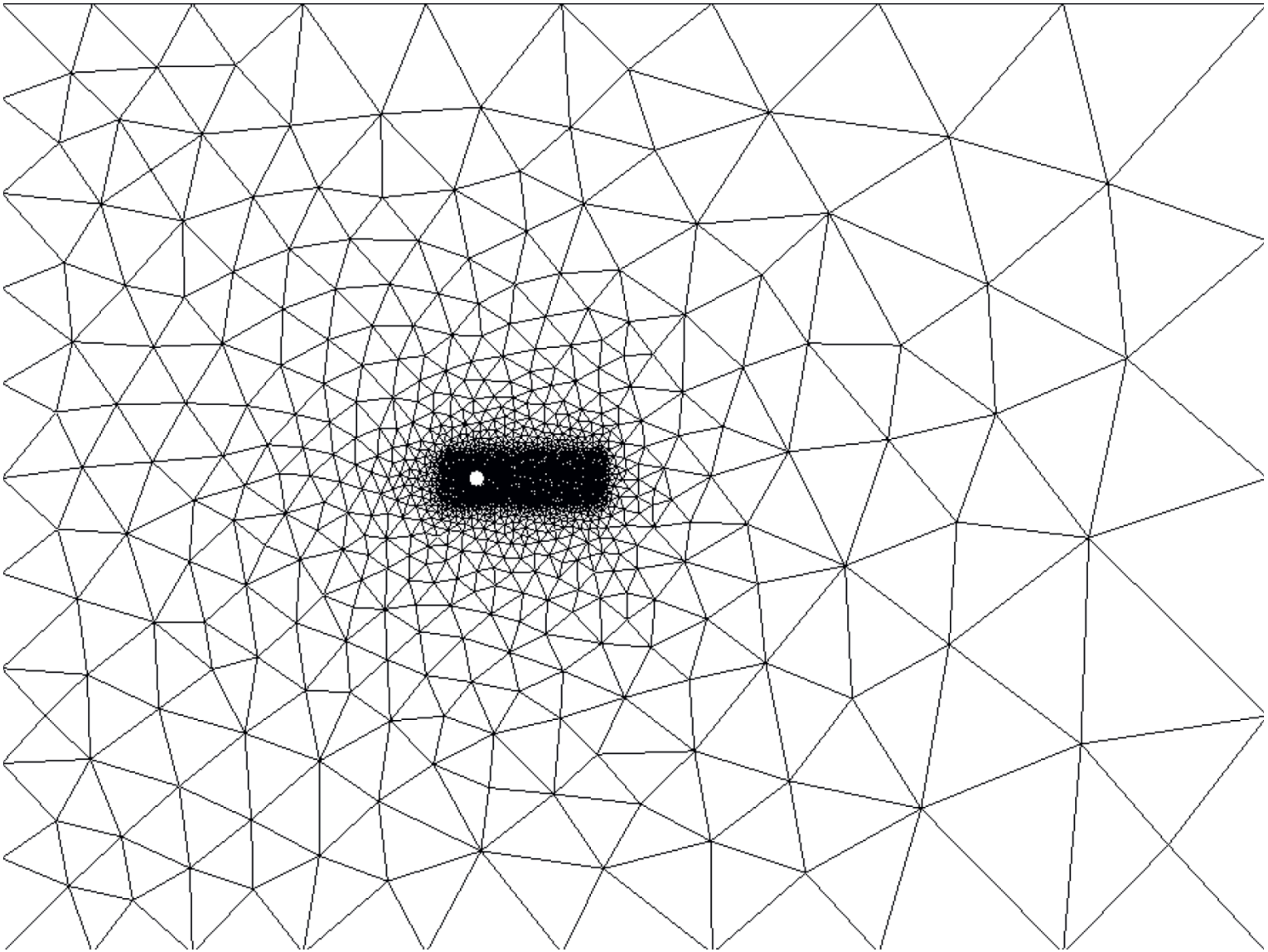
Robustness in 2D: Ma = 7.7e-2, Re = 1e6, P=4



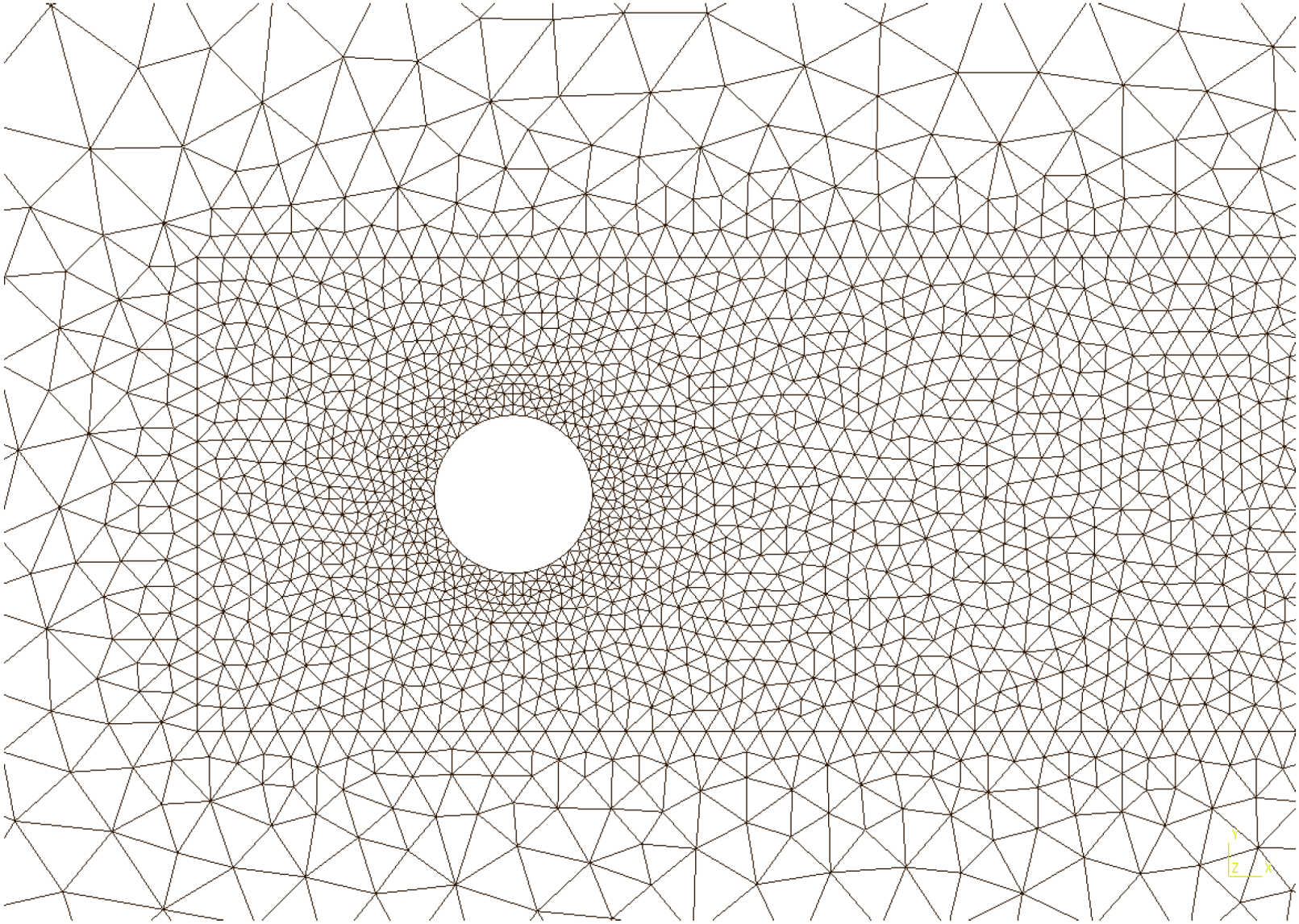
Robustness in 2D: Ma = 1.5, Re = 1e6, P=4



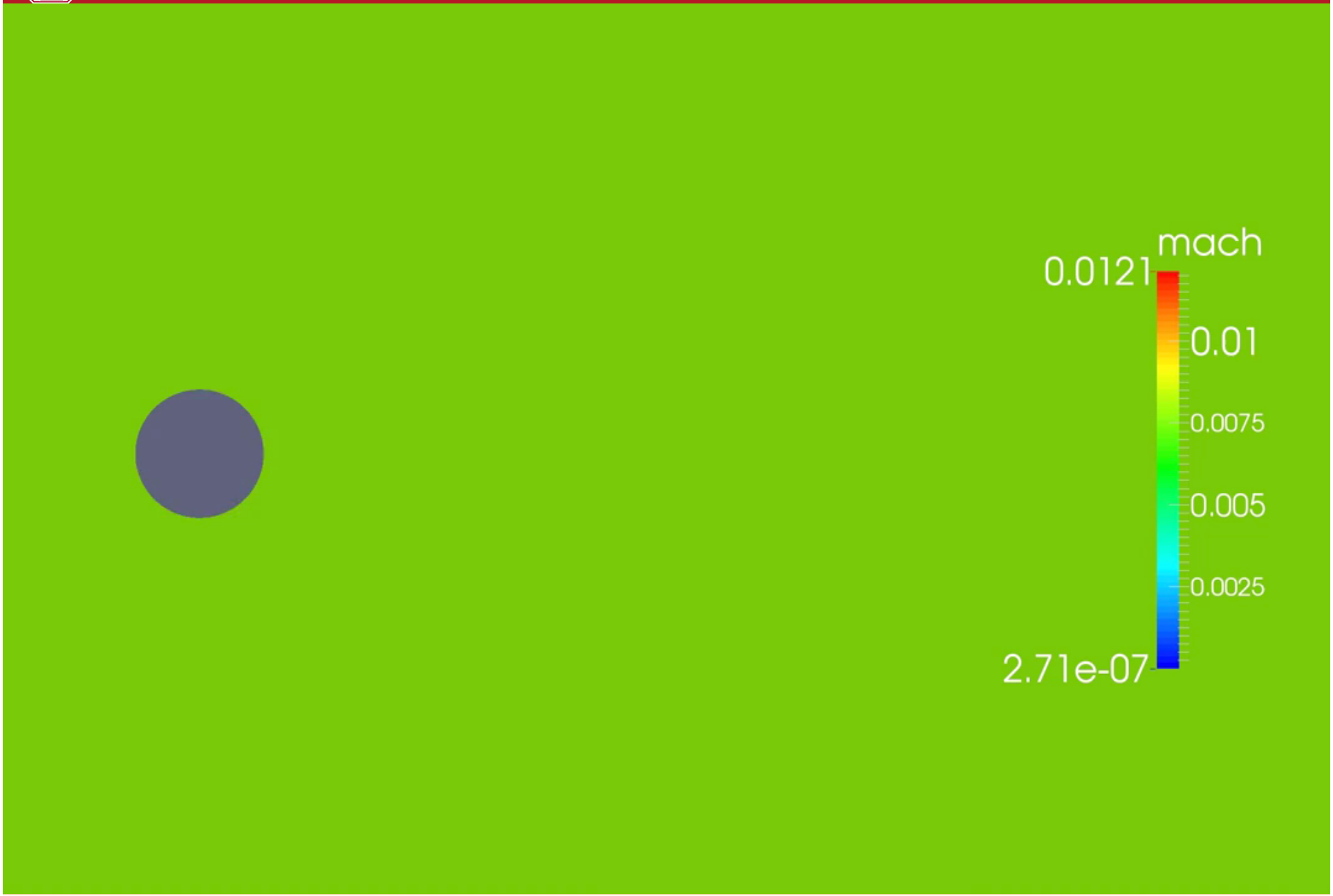
Robustness in 2D: Ma = 7.7e-3, Re = 1e6



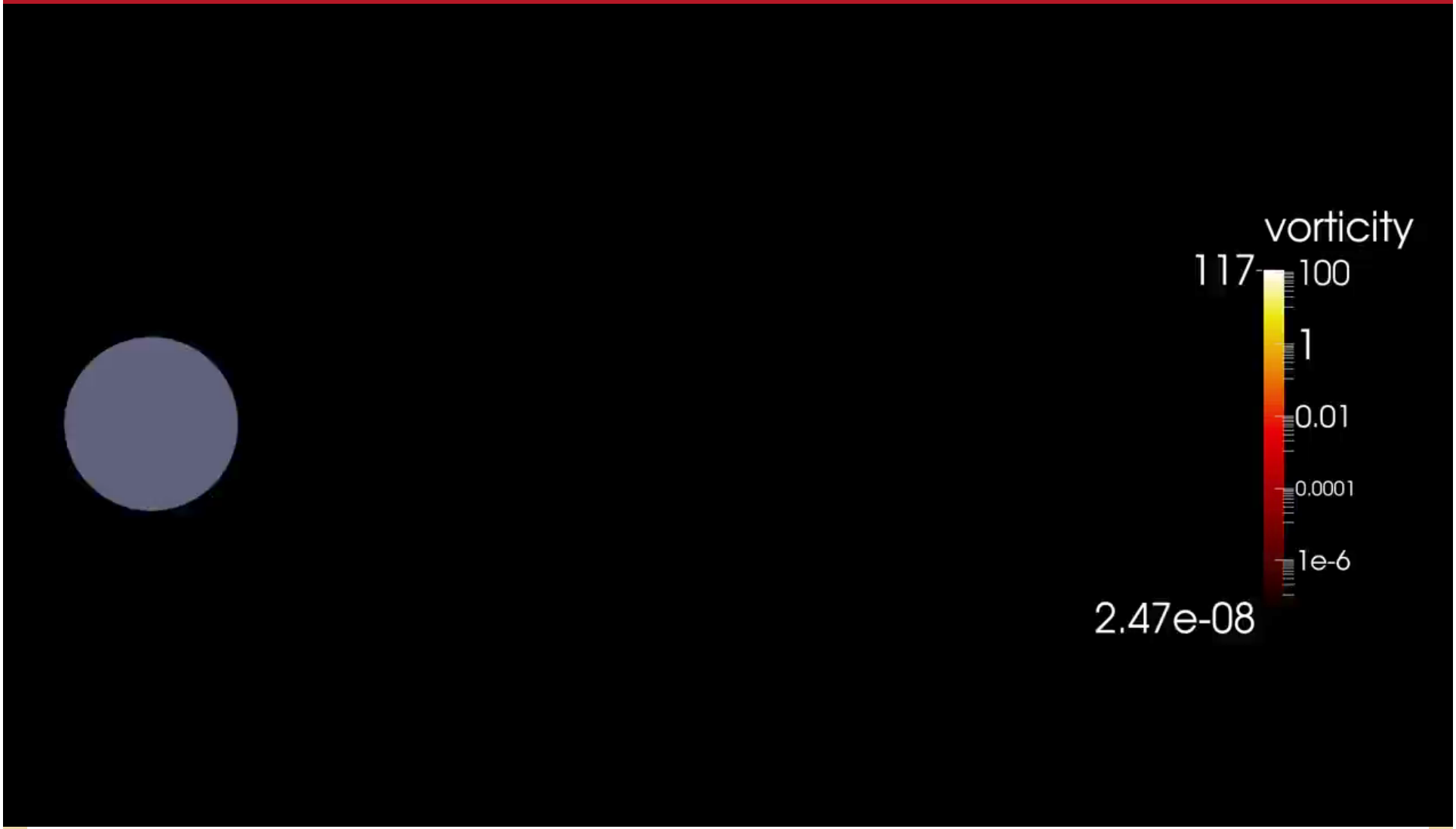
Robustness in 2D: Ma = 7.7e-3, Re = 1e6



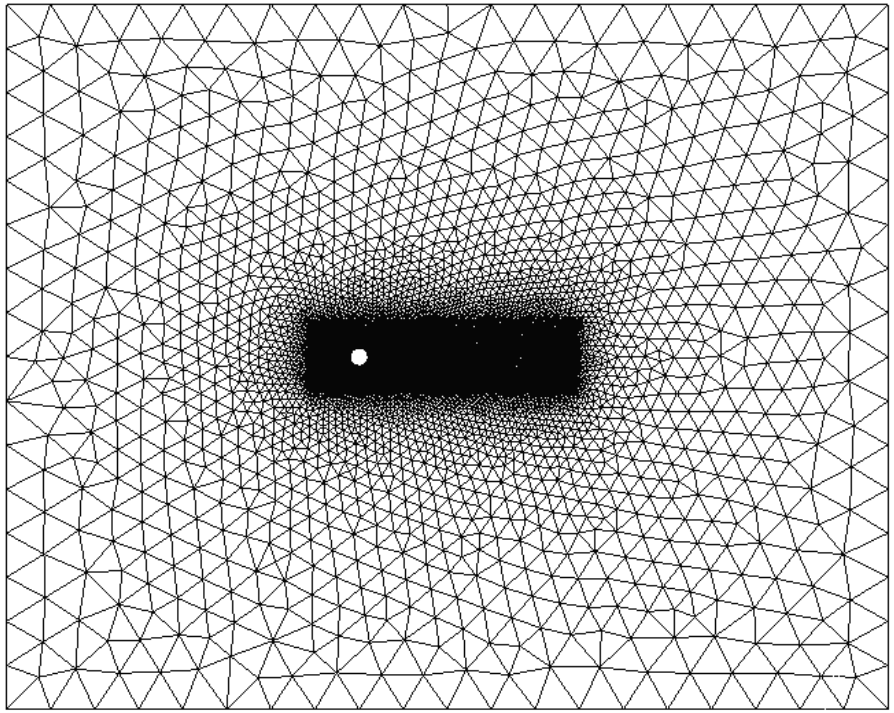
Robustness in 2D: Ma = 7.7e-3, Re = 1e6, P=4

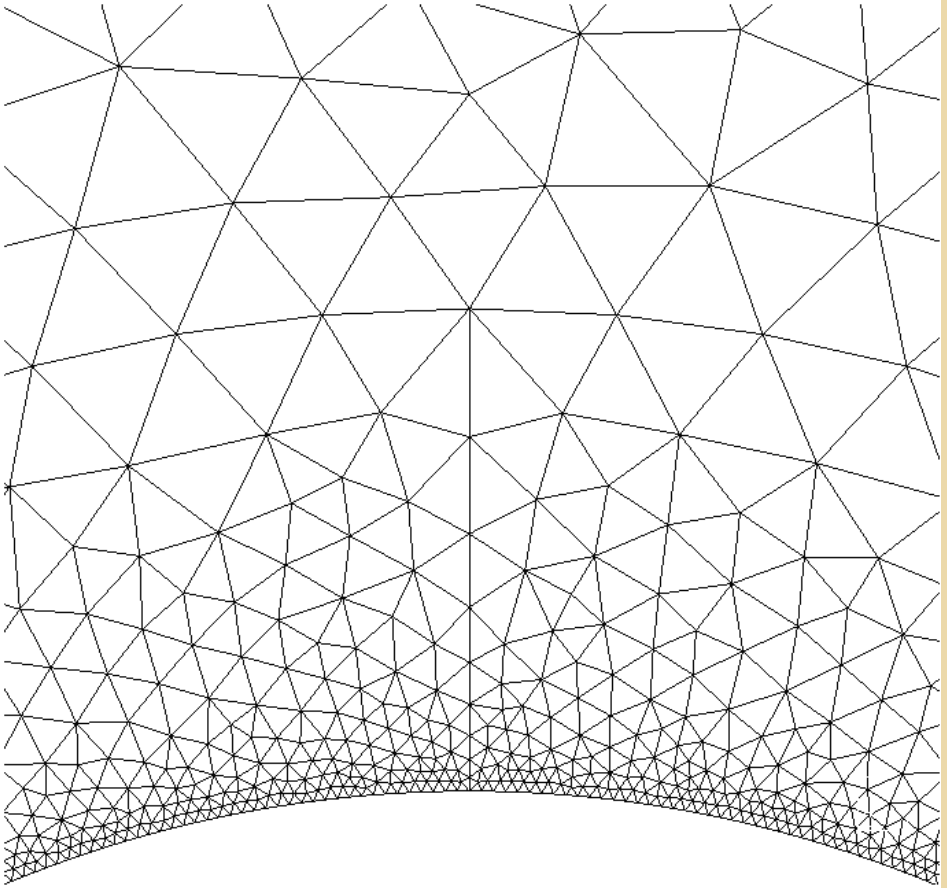


Robustness in 2D: Ma = 7.7e-3, Re = 1e6, P=4

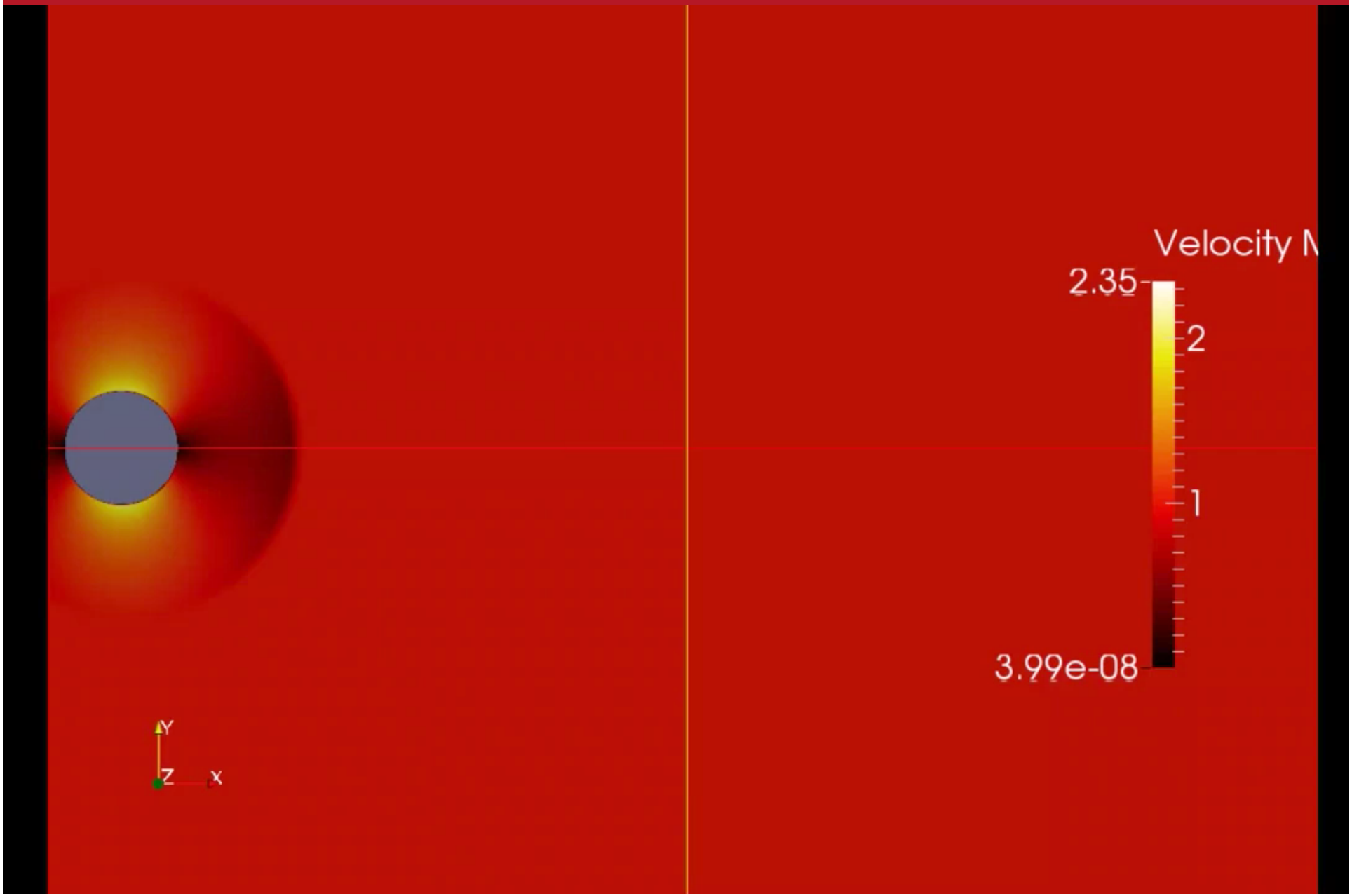


Effects on flow 2D: Ma = 0.1, Re = 3.9e3

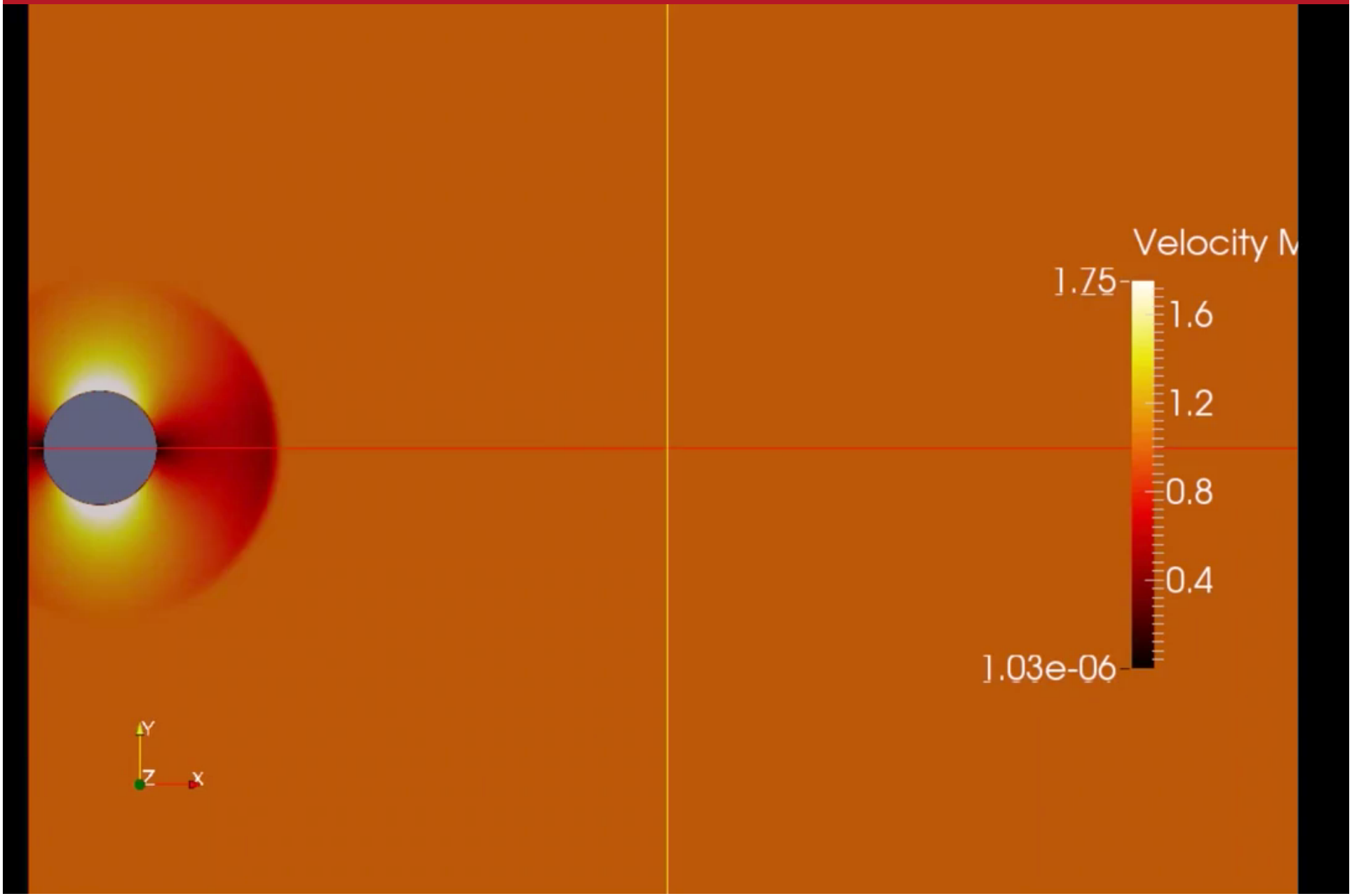




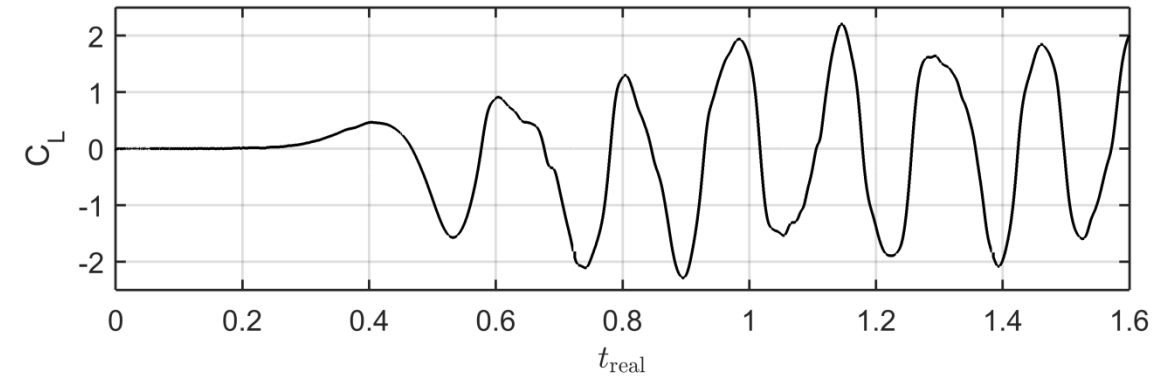
Ma = 0.1, Re = 3.9e3, P = 4, Unfiltered

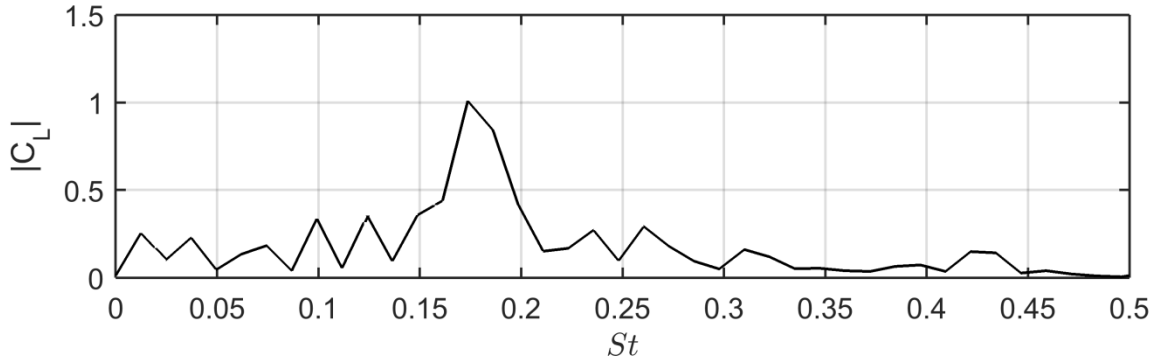


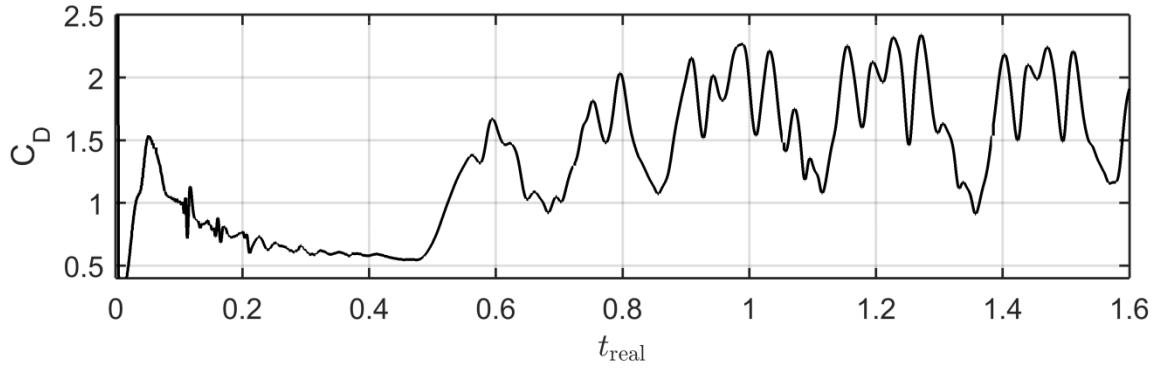
Ma = 0.1, Re = 3.9e3, P = 4, Filtered



Ma = 0.1, Re = 3.9e3, P = 4, Unfiltered

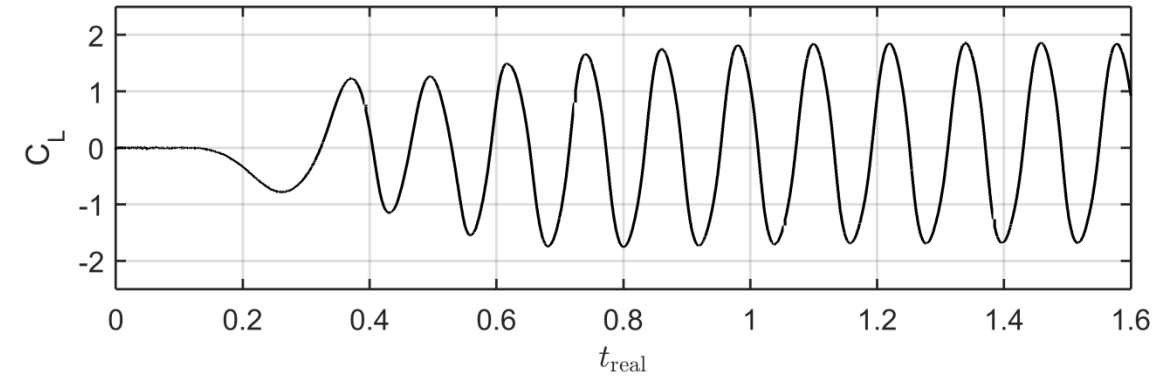


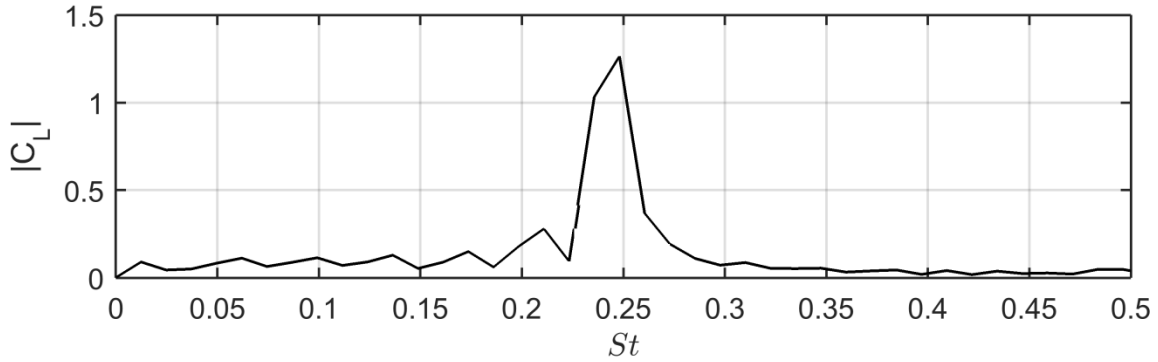


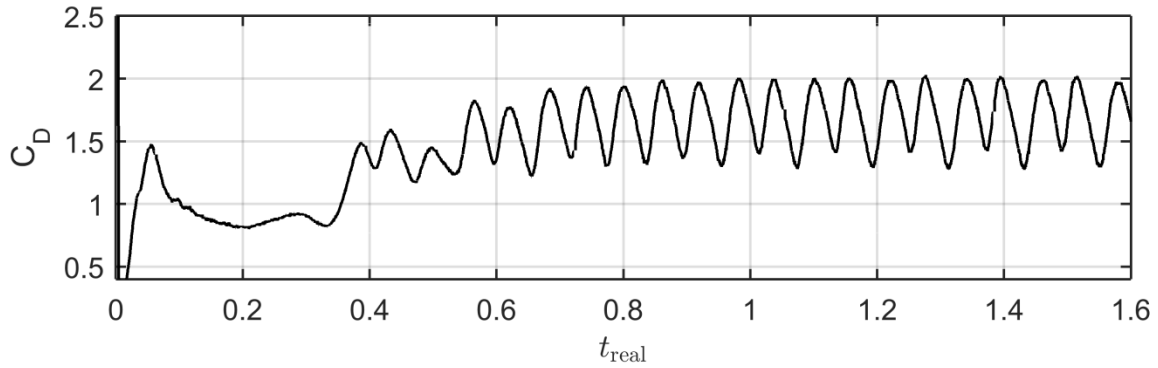


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Ma = 0.1, Re = 3.9e3, P = 4, Filtered







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Conclusion

- Developed filters with physical interpretation for general elements
  - compact stencil
  - performed via small matrix multiplication
  - implemented in GPUs
- Stabilized extreme simulation cases with little to no tuning
- De-coupled robustness from grid quality
- Implementation shared on hifiles.stanford.edu, in LFS-filters branch



Future Work

- Implement filters on tetrahedral elements
- Estimate minimal filtering needed
- Assess usefulness as pre-conditioner: boundary conditions are satisfied



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Questions?

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