How wrong can a flow model be, and yet provide a reasonable acoustic prediction ?

Christophe Schram, Associate Professor

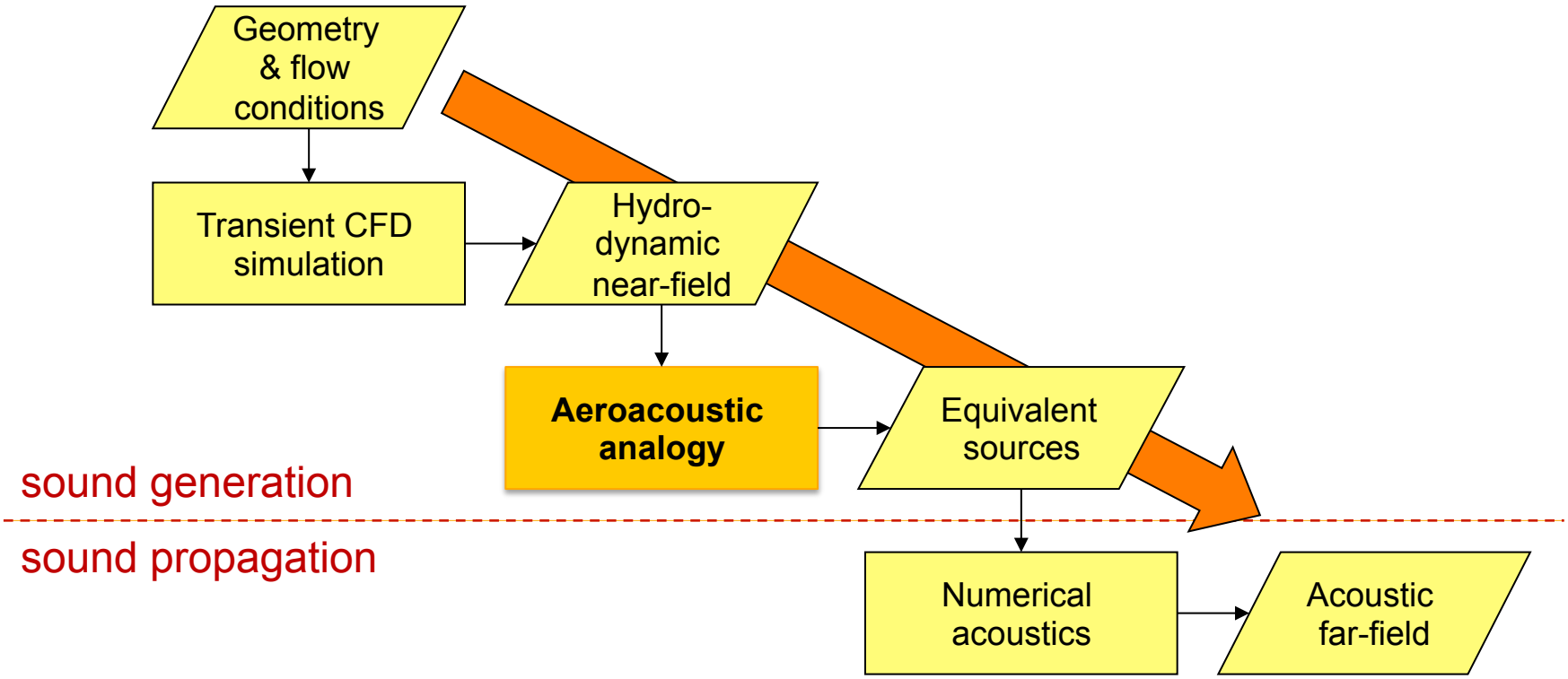
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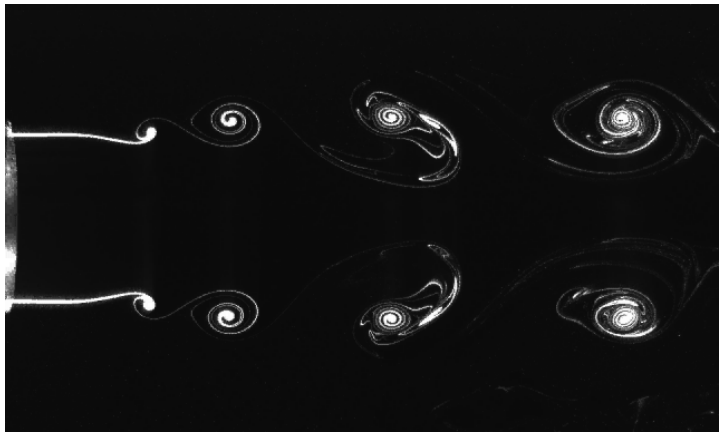
NASA Ames Research Center – July 1st, 2014 – christophe.schram@vki.ac.be

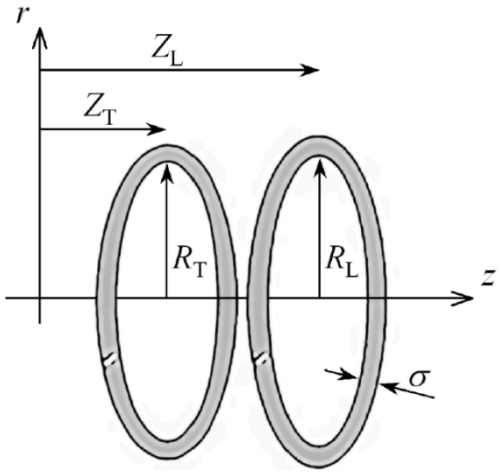
A primitive approach to aeroacoustics…



… and two questions

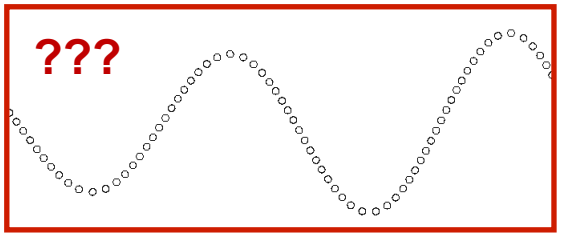
First question: why does a 2D model of periodic vortex pairing give linearly increasing pressure fluctuations ?





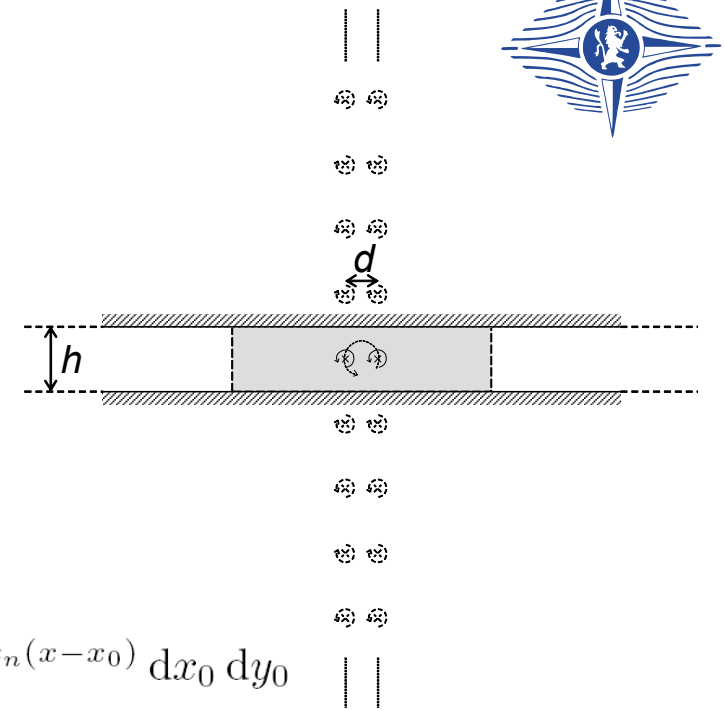
$$
\begin{aligned}
Z_L(t) &= u_0 t + \frac{d}{2}\cos\left(\frac{\Gamma t}{\pi d^2}\right)\\
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R_T(t) &= R_0 - \frac{d}{2}\sin\left(\frac{\Gamma t}{\pi d^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
p'(\mathbf{x},t) = \frac{\rho_0}{4c_0^2|\mathbf{x}|^3}\Bigg\{\Bigg[\Bigg(&-\frac{4\Gamma^4 R_0}{\pi^3 d^4}+\frac{3\Gamma^3 u_0}{\pi^2 d^2}\Bigg)\cos\left(\frac{2\Gamma t}{\pi^2 d^2}\right)\\
&-\frac{2\Gamma^4 u_0}{\pi^3 d^4}\,t\,\sin\left(\frac{2\Gamma t}{\pi^2 d^2}\right)\Bigg]\,\mathbf{x}\cdot(\mathbf{n}\,\mathbf{n})\cdot\mathbf{x}\\
&+\left(\frac{2\Gamma^4 R_0}{\pi^3 d^4}-\frac{\Gamma^3 u_0}{\pi^2 d^2}\right)\cos\left(\frac{2\Gamma t}{\pi^2 d^2}\right)(\mathbf{x}\cdot\mathbf{x})\Bigg\}
\end{aligned}
$$



… and two questions

Second question: why does Curle's analogy <u>always</u> give me wrong results when I'm using an incompressible model for non-compact ducted flows ?

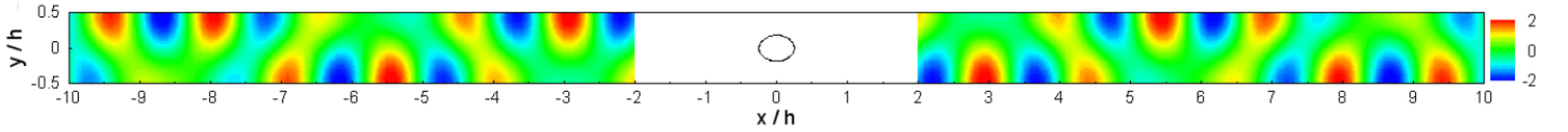


$$p_{\mathrm{a}}(\mathbf{x},\omega) = -\frac{\mathrm{i}}{2h}\sum_{n=0}^{\infty}\frac{\cos(\eta_n y)}{k_n C_n}$$

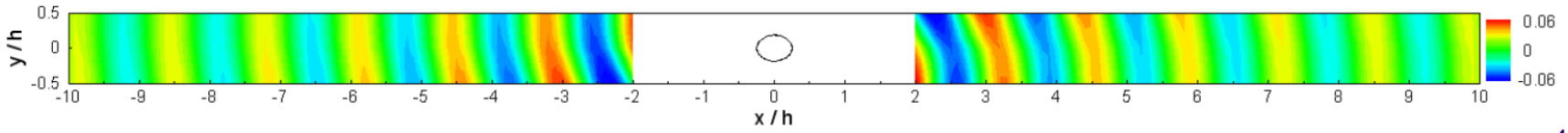
Exact solution (based on tailored Green's function):

$$\left\{ k_n^2 \iint_{S_0} \cos(\eta_n y_0)\,\rho_0 u^2\,\mathrm{e}^{\mp \mathrm{i}k_n(x-x_0)}\,\mathrm{d}x_0\,\mathrm{d}y_0 \right.$$
$$+ \eta_n^2 \iint_{S_0} \cos(\eta_n y_0)\,\rho_0 v^2\,\mathrm{e}^{\mp \mathrm{i}k_n(x-x_0)}\,\mathrm{d}x_0\,\mathrm{d}y_0$$
$$\left. \pm \mathrm{i}k_n\eta_n \iint_{S_0} \sin(\eta_n y_0)\,\rho_0 uv\,\mathrm{e}^{\mp \mathrm{i}k_n(x-x_0)}\,\mathrm{d}x_0\,\mathrm{d}y_0 \right\}$$

Exact:



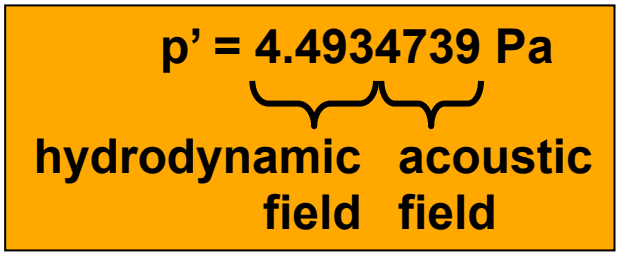
Curle:



Low Mach number issues

- Aeroacoustic analogies show a large sensitivity to seemingly small approximations in the flow model
- Blame it on aeroacoustic analogies ?

- At low Mach numbers: orders of magnitude of difference between hydrodynamic and acoustic fluctuations (O(M^4))

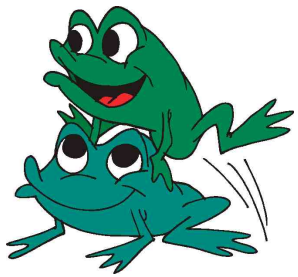


- Direct Noise Computations (DNC) → additional issues (dissipation/dispersion → numerical cost $\propto Re^2M^{-4}$, NRBCs, synthetic turbulence generation, …)

- Are there better choices to be made in the decoupling between sound generation and propagation effects ?
- **Are there aeroacoustic analogies that perform better than others ?**

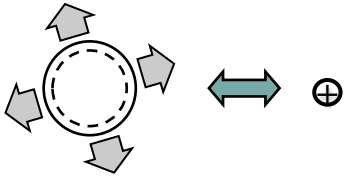
Plan

- Fundamentals of Lighthill's acoustic analogy
  - Concepts
  - Approximations
- First question: sound emitted by free vortex leapfrogging
  - Review of Vortex Sound Theory
  - Predictions based on analytical model and experimental data
- Second question: sound emitted by ducted vortex leapfrogging, based on analytical model
  - Review of Curle's analogy
  - The black magic behind the acoustic dipoles



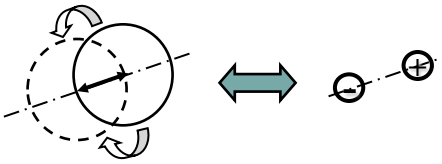
Monopoles, dipoles, quadrupoles

- Monopole = pulsating sphere
- Physically: unsteady combustion, pipe exhaust, vocal folds, …



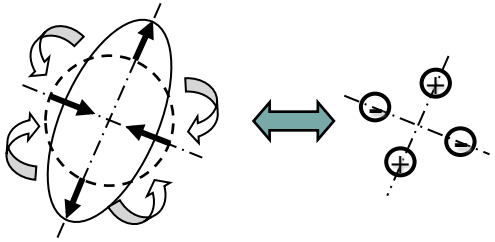
$$p' \propto (ka)^2$$

- Dipole = oscillating sphere without change of volume
- Less efficient than monopole
- Physically: unsteady forces



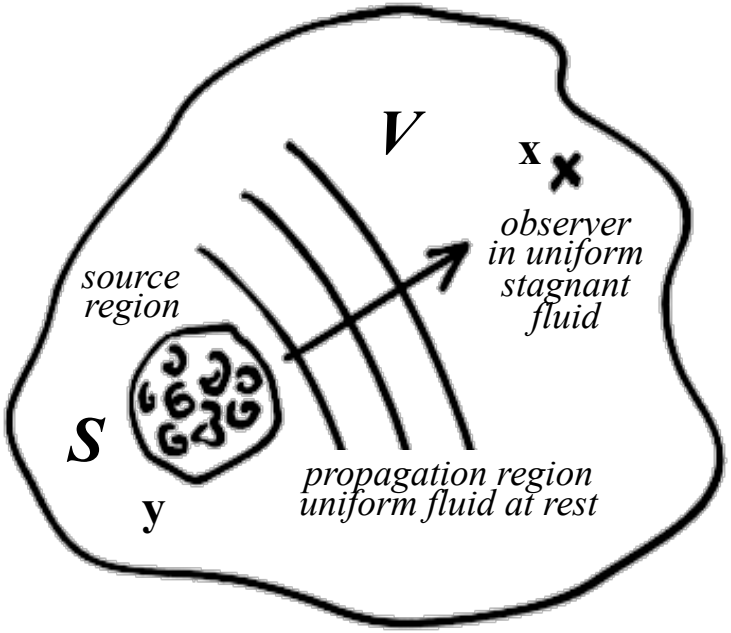
$$p' \propto (ka)^3$$

- Quadrupole = deforming sphere without change of volume nor net force
- Less efficient than dipole
- Physically: turbulence



$$p' \propto (ka)^4$$

Lighthill's aeroacoustical analogy : concept



- The problem of sound produced by a turbulent flow is, **from the listener's point of view**, analogous to a problem of propagation in a uniform medium at rest in which equivalent sources are placed.
- Wave propagation region: linear wave operator applies

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \nabla^2 \rho' = q(\mathbf{x}, t)$$

- Turbulent region: fluid mechanics equations apply

mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

momentum

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = -\nabla p + \nabla \boldsymbol{\sigma} + \cancel{\mathbf{f}}$$

no external forces
→ no dipoles

Lighthill's analogy: definition of a reference state

- Reformulation of fluid mechanics equations, and use of arbitrary speed c_0 :

$$\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \frac{\partial^2 \rho}{\partial x_i^2} = \frac{\partial^2 (\rho v_i v_j - \sigma_{ij})}{\partial x_i \partial x_j} + \frac{\partial^2 (p - c_0^2 \rho)}{\partial x_i^2}$$

- Definition of a reference state:

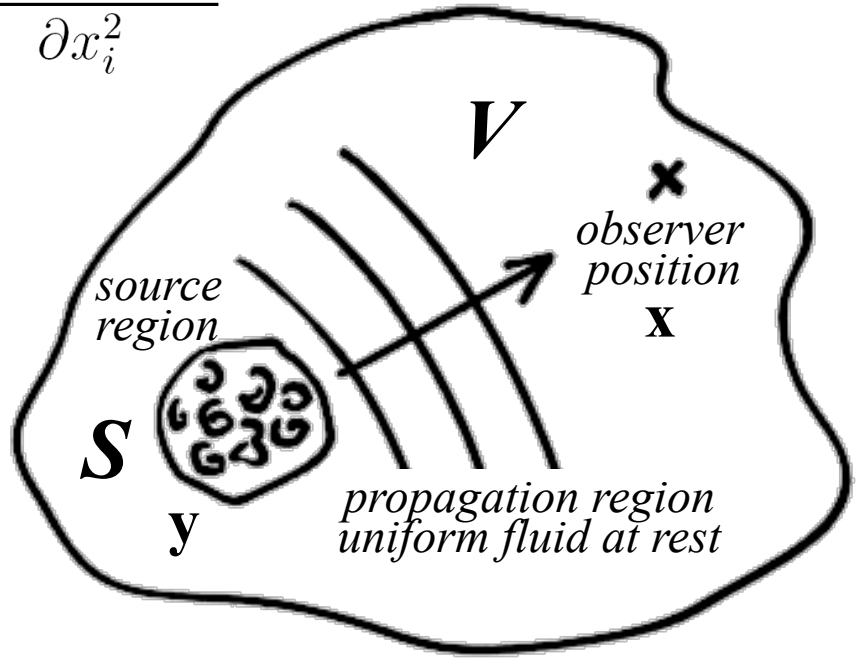
$$\rho' \equiv \rho - \rho_0$$
$$p' \equiv p - p_0$$
$$v_i' \equiv v_i$$

- Aeroacoustical analogy :

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$$

with $T_{ij} = \rho v_i v_j + (p' - c_0^2 \rho') \delta_{ij} - \sigma_{ij}$

Lighthill's tensor



Exact… therefore perfectly useless!

Sound produced by free isothermal turbulent flows at low Mach number: some useful approximations

- Solution using Green's fct

$$\rho'(\mathbf{x},t) = \int_{-\infty}^{t} \iiint_V \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} G \, \mathrm{d}^3\mathbf{y} \mathrm{d}\tau - c_0^2 \int_{-\infty}^{t} \iint_{\partial V} \left(\rho' \frac{\partial G}{\partial y_i} - G \frac{\partial \rho'}{\partial y_i} \right) n_i \, \mathrm{d}^2\mathbf{y} \, \mathrm{d}\tau$$

sound scattering at boundaries

Integral solution → good for numerical stability

- Purpose: simplify the RHS $\dfrac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$

no entropy generation
→ no monopoles

$$T_{ij} = \rho v_i v_j + (p' - c_0^2 \rho') \delta_{ij} - \sigma_{ij}$$

  - High Reynolds number
  - Isentropic
  - Low Mach number → $\rho v_i v_j \simeq \rho_0 v_i v_j$

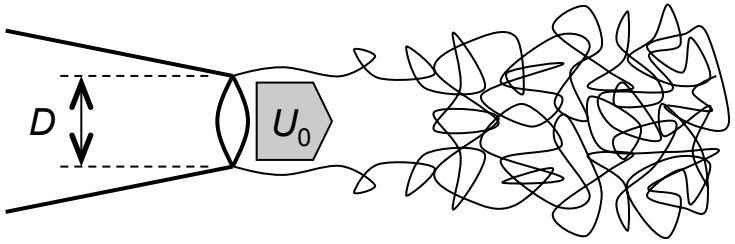
Explicit integral solution → decoupling between source and propagation effects, allows using incompressible flow model

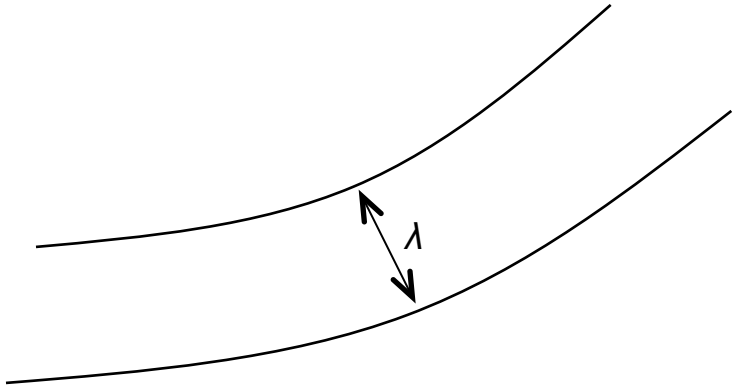
Lighthill's M^8 law for low Mach number free jets

- Using free field Green's function: $$\rho'(\mathbf{x},t) = \frac{\partial^2}{\partial x_i \partial x_j} \iiint_V \left[\frac{\rho_0 v_i v_j}{4\pi c_0^2 |\mathbf{x}-\mathbf{y}|} \right] \mathrm{d}^3\mathbf{y}$$

$$\hookrightarrow t^* = t - |\mathbf{x}-\mathbf{y}|/c_0$$

- Scaling law:





Acoustic scale: $x \propto \lambda = c_0/f$

Flow time scale: D/U_0

Spatial derivative: $U_0/(c_0 D)$

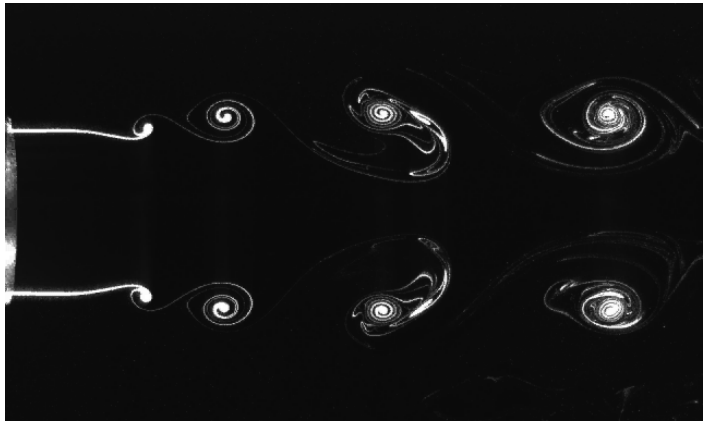
$$\Rightarrow p' = c_0^2 \rho' \propto \frac{U_0^2}{c_0^2 D^2} \frac{\rho_0 U_0^2 D^3}{|\mathbf{x}|} = \rho_0 c_0^2 M^4 \frac{D}{|\mathbf{x}|}$$

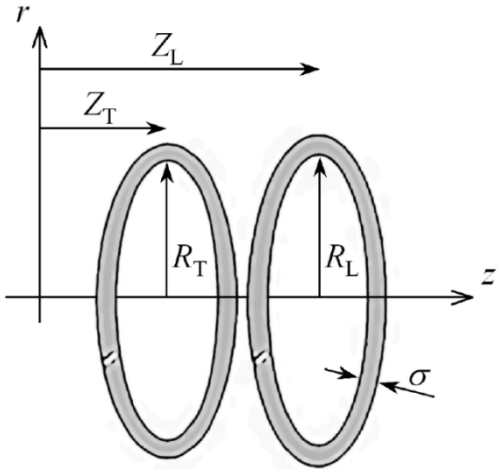
$$\Rightarrow \text{Acoustical power: } W = \frac{4\pi |\mathbf{x}|^2 p'^2}{\rho_0 c_0} \propto \rho_0 c_0^3 D^2 M^8$$

Quadrupolar source

Not by chance! We obtain quadrupolar radiation efficiency because we imposed it!

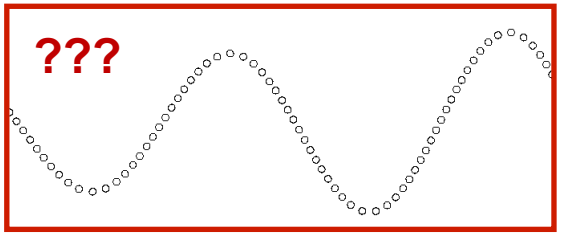
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Z_{\mathrm{T}}(t) &= u_0 t - \frac{d}{2}\cos\left(\frac{\Gamma t}{\pi d^2}\right)\\
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&-\frac{2\Gamma^4 u_0}{\pi^3 d^4}\,t\,\sin\left(\frac{2\Gamma t}{\pi^2 d^2}\right)\Bigg]\,\mathbf{x}\cdot(\mathbf{n}\,\mathbf{n})\cdot\mathbf{x}\\
&+\left(\frac{2\Gamma^4 R_0}{\pi^3 d^4}-\frac{\Gamma^3 u_0}{\pi^2 d^2}\right)\cos\left(\frac{2\Gamma t}{\pi^2 d^2}\right)(\mathbf{x}\cdot\mathbf{x})\Bigg\}
\end{aligned}
$$



Vortex Sound Theory: Powell's analogy

- Vectorial identity: $\nabla\left(\frac{|\mathbf{v}|^2}{2}\right) = \mathbf{v}\times\boldsymbol{\omega} + \mathbf{v}\cdot\nabla\mathbf{v}$
- Momentum equation becomes: $\rho\frac{\partial\mathbf{v}}{\partial t} + \rho\nabla\left(\frac{1}{2}|\mathbf{v}|^2\right) + \rho(\boldsymbol{\omega}\times\mathbf{v}) + \nabla p = 0$
- Similar manipulation as for Lighthill's analogy:

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = \nabla\cdot\left[\rho(\boldsymbol{\omega}\times\mathbf{v}) + \nabla\left(\frac{1}{2}\rho|\mathbf{v}|^2\right) - \mathbf{v}\frac{\partial\rho}{\partial t} - \frac{1}{2}|\mathbf{v}|^2\nabla\rho\right] + \frac{\partial^2}{\partial t^2}\left(\frac{p'}{c_0^2} - \rho'\right)$$

low Mach; isentropic; $\propto M^2$

- Retaining leading order terms in M^2:

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = \nabla\cdot[\rho(\boldsymbol{\omega}\times\mathbf{v})] + \nabla^2\left(\frac{1}{2}\rho|\mathbf{v}|^2\right)$$

conservation of kinetic energy

- Integral solution using free field Green's function and first order Taylor expansion of the retarded time:

$$p'(\mathbf{x},t) = -\frac{x_i}{4\pi c_0|\mathbf{x}|^2}\frac{\partial}{\partial t}\iiint_V \rho(\boldsymbol{\omega}\times\mathbf{v})_i\,\mathrm{d}^3\mathbf{y} - \frac{x_i x_j}{4\pi c_0^2|\mathbf{x}|^3}\frac{\partial^2}{\partial t^2}\iiint_V y_j\,\rho(\boldsymbol{\omega}\times\mathbf{v})_i\,\mathrm{d}^3\mathbf{y}$$

conservation of momentum

- Powell's integral formulation: $p'_{\mathrm{P}}(\mathbf{x},t) = -\frac{\rho_0}{4\pi c_0^2|\mathbf{x}|^3}\frac{\partial^2}{\partial t^2}\iiint_V (\mathbf{x}\cdot\mathbf{y})\,\mathbf{x}\cdot(\boldsymbol{\omega}\times\mathbf{v})\,\mathrm{d}^3\mathbf{y}$

Vortex Sound Theory: Möhring's analogy

- Starting from Powell's integral formulation:

$$p'_{\mathrm{P}}(\mathbf{x},t) = -\frac{\rho_0}{4\pi c_0^2|\mathbf{x}|^3}\frac{\partial^2}{\partial t^2}\iiint_V (\mathbf{x}\cdot\mathbf{y})\,\mathbf{x}\cdot(\boldsymbol{\omega}\times\mathbf{v})\,\mathrm{d}^3\mathbf{y}$$

- Using vectorial's identity:

$$\nabla_y \times \left[\frac{1}{3}(\mathbf{x}\cdot\mathbf{y})\,\mathbf{x}\times\mathbf{y}\right] = (\mathbf{x}\cdot\mathbf{y})\,\mathbf{x} - \frac{1}{3}|\mathbf{x}|^2\mathbf{y}$$

- By substitution:

$$p'(\mathbf{x},t) = -\frac{\rho_0}{12\pi c_0^2|\mathbf{x}|^3}\frac{\partial^2}{\partial t^2}\left\{\iiint_V \nabla_y\times[(\mathbf{x}\cdot\mathbf{y})\,\mathbf{x}\times\mathbf{y}]\cdot(\boldsymbol{\omega}\times\mathbf{v})\,\mathrm{d}^3\mathbf{y} + |\mathbf{x}|^2\iiint_V \mathbf{y}\cdot(\boldsymbol{\omega}\times\mathbf{v})\,\mathrm{d}^3\mathbf{y}\right\}$$

conservation of kinetic energy

- Using Helmholtz's vorticity transport equation: $\dfrac{\partial\boldsymbol{\omega}}{\partial t} + \nabla_y\times(\boldsymbol{\omega}\times\mathbf{v}) = 0$

- Möhring's integral formulation:

$$p'_{\mathrm{M}}(\mathbf{x},t) = \frac{\rho_0}{12\pi c_0^2|\mathbf{x}|^3}\frac{\partial^3}{\partial t^3}\iiint_V (\mathbf{x}\cdot\mathbf{y})\,\mathbf{x}\cdot(\mathbf{y}\times\boldsymbol{\omega})\,\mathrm{d}^3\mathbf{y}$$

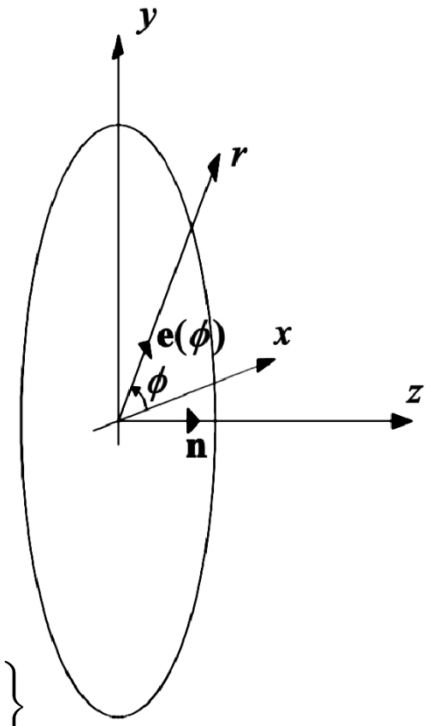
Vortex Sound Theory:
2 solutions for the same problem



- We have derived two (formally) equivalent formulations of the Vortex Sound Theory:
  - Powell's analogy: $$p'_{\mathrm{P}}(\mathbf{x},t) = -\frac{\rho_0}{4\pi c_0^2|\mathbf{x}|^3}\frac{\partial^2}{\partial t^2}\iiint_V (\mathbf{x}\cdot\mathbf{y})\,\mathbf{x}\cdot(\boldsymbol{\omega}\times\mathbf{v})\,\mathrm{d}^3\mathbf{y}$$
  - Mohring's analogy: $$p'_{\mathrm{M}}(\mathbf{x},t) = \frac{\rho_0}{12\pi c_0^2|\mathbf{x}|^3}\frac{\partial^3}{\partial t^3}\iiint_V (\mathbf{x}\cdot\mathbf{y})\,\mathbf{x}\cdot(\mathbf{y}\times\boldsymbol{\omega})\,\mathrm{d}^3\mathbf{y}$$
- Although formally equivalent, these two formulations do not yield the same numerical robustness!

Vortex Sound Theory for axisymmetrical flows

- Coordinate of a vortex element: $\mathbf{y} = z\,\mathbf{n} + r\,\mathbf{e}(\phi)$



- General form of velocity and vorticity:

$$\begin{cases} \boldsymbol{\omega}(r,\phi,z) &= \omega(r,z)\,\mathbf{n}\times\mathbf{e}(\phi) \\ \mathbf{v}(r,\phi,z) &= u(r,z)\,\mathbf{n} + v(r,z)\,\mathbf{e}(\phi) + w(r,z)\,\mathbf{n}\times\mathbf{e}(\phi) \end{cases}$$

- Powell's analogy becomes:

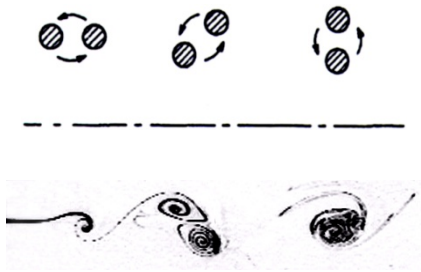
$$p'_{\mathrm{P}}(\mathbf{x},t) = \frac{\rho_0}{4c_0^2|\mathbf{x}|^3}\frac{\partial^2}{\partial t^2}\left\{2\left(\iint_S \omega v r z\,\mathrm{d}r\mathrm{d}z\right)(\mathbf{x}\cdot\mathbf{n})(\mathbf{x}\cdot\mathbf{n}) - \left(\iint_S \omega u r^2\,\mathrm{d}r\mathrm{d}z\right)[(\mathbf{x}\cdot\mathbf{x}) - (\mathbf{x}\cdot\mathbf{n})(\mathbf{x}\cdot\mathbf{n})]\right\}$$

- Möhring's analogy becomes:

$$p'_{\mathrm{M}}(\mathbf{x},t) = \frac{\rho_0}{4c_0^2|\mathbf{x}|^3}\frac{\mathrm{d}^3Q}{\mathrm{d}t^3}\,\mathbf{x}\cdot\left(\mathbf{n}\,\mathbf{n} - \frac{\mathbf{I}}{3}\right)\cdot\mathbf{x} \qquad Q = \iint_S \omega r^2 z\,\mathrm{d}r\mathrm{d}z$$

Vortex ring pairing

- Vortex pairing = inviscid interaction (Biot-Savart)
  - Vortex leapfrogging: periodic motion
  - Vortex merging: requires core deformation



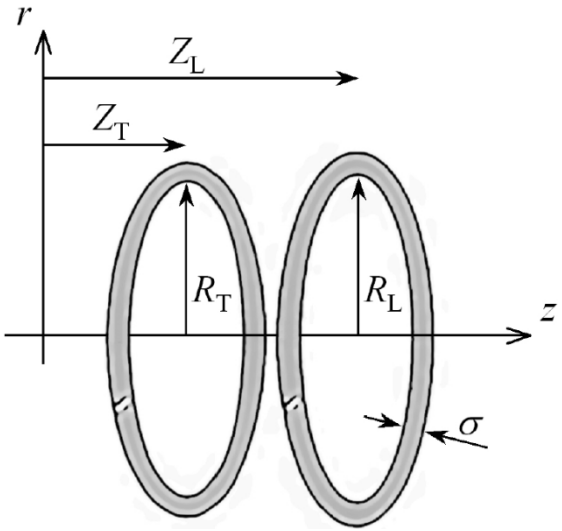
- Can be easily stabilized and studied at laboratory scale
- One of the mechanisms of sound production in subsonic jets

- Sound prediction based on two different source models:
  - Analytical models (2D and 3D axisymmetric)
    → controllable error
  - Experimental data: time-resolved Particle Image Velocimetry
    → random error

2D and 3D models of vortex ring leapfrogging

- 2D model ($\sigma << d << R_0$): locally planar interaction, neglects vortex stretching

$$\begin{cases} Z_{\mathrm{L}}(t) &= u_0 t + \dfrac{d}{2}\cos\left(\dfrac{\Gamma t}{\pi d^2}\right) \\ R_{\mathrm{L}}(t) &= R_0 + \dfrac{d}{2}\sin\left(\dfrac{\Gamma t}{\pi d^2}\right) \\ Z_{\mathrm{T}}(t) &= u_0 t - \dfrac{d}{2}\cos\left(\dfrac{\Gamma t}{\pi d^2}\right) \\ R_{\mathrm{T}}(t) &= R_0 - \dfrac{d}{2}\sin\left(\dfrac{\Gamma t}{\pi d^2}\right) \end{cases}$$



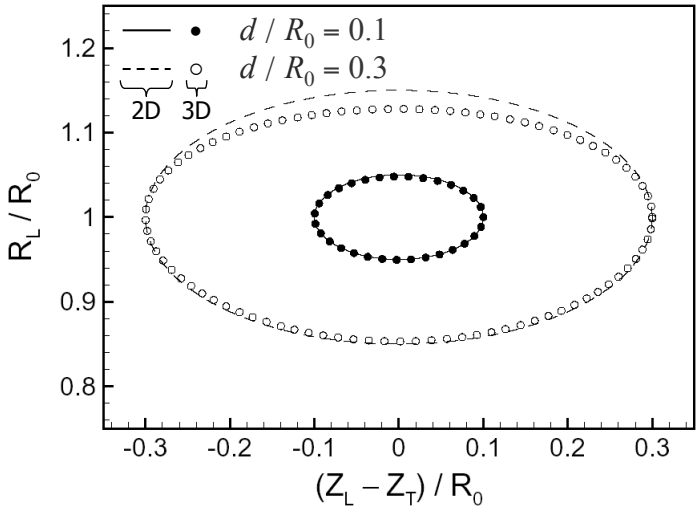
- 3D model ($\sigma << d = \mathrm{O}(R_0)$): accounts for vortex stretching

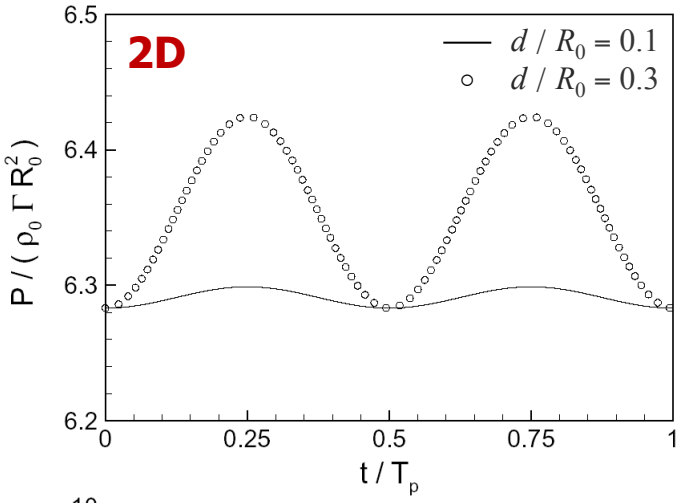
$$\begin{cases} \dfrac{\mathrm{d}Z_m}{\mathrm{d}t} &= \dfrac{1}{R_m}\dfrac{\partial \Psi}{\partial R_m} + \dfrac{\Gamma_m}{4\pi R_m}\left[\log\left(\dfrac{8R_m}{\sigma_{c,m}}\right) - \dfrac{1}{4}\right] \\ \dfrac{\mathrm{d}R_m}{\mathrm{d}t} &= -\dfrac{1}{R_m}\dfrac{\partial \Psi}{\partial Z_m} \\ \dfrac{\mathrm{d}\sigma_{c,m}^2 R_m}{\mathrm{d}t} &= 0 \end{cases}$$

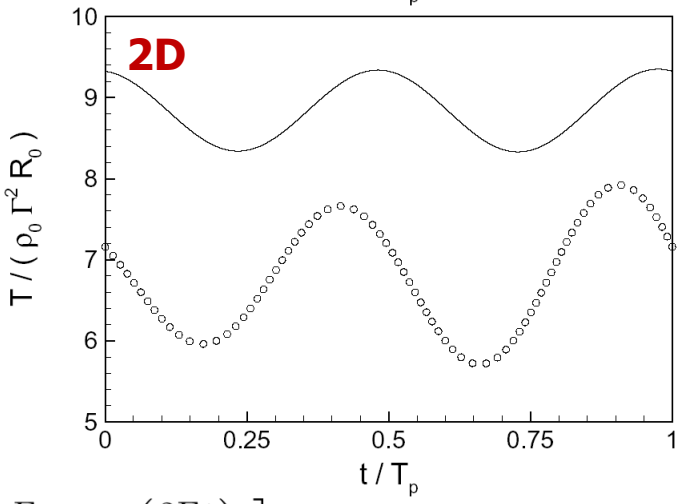
$$\Psi = \frac{\Gamma_n}{2\pi}\sqrt{R_m R_n}\left[\left(\frac{2}{k_{mn}} - k_{mn}\right)K(k_{mn}) - \frac{2}{k_{mn}}E(k_{mn})\right] \qquad k_{mn} = \sqrt{\frac{4R_m R_n}{(Z_m - Z_n)^2 + (R_m + R_n)^2}}$$

2D model: vortex trajectories and flow invariants

- Two cases considered: $d / R_0 = 0.1$ and 0.3
- Locus of the vortex cores:







- Flow invariants:

$$P = 2\pi\rho_0\Gamma R_0^2 \left[1 + \frac{d^2}{4R_0^2}\sin^2\left(\frac{\Gamma t}{\pi d^2}\right)\right]$$

$$T = 2\pi\rho_0\Gamma \left[2R_0^2 u_0 - \frac{\Gamma R_0}{2\pi} + \left(\frac{u_0 d^2}{2} - \frac{\Gamma R_0}{2\pi}\right)\sin^2\left(\frac{\Gamma t}{\pi d^2}\right) - \frac{u_0\Gamma}{4\pi}\, t\, \sin\left(\frac{2\Gamma t}{\pi d^2}\right)\right]$$

secular term

2D model: sound prediction

- Powell's analogy:

$$p'_{\mathrm{P}}(\mathbf{x},t) = \frac{\rho_0}{4c_0^2|\mathbf{x}|^3}\left\{\left[\left(-\frac{4\Gamma^4 R_0}{\pi^3 d^4}+\frac{3\Gamma^3 u_0}{\pi^2 d^2}\right)\cos\left(\frac{2\Gamma t}{\pi^2 d^2}\right)\right.\right.$$
$$\left.\left.\text{secular term}\quad -\frac{2\Gamma^4 u_0}{\pi^3 d^4}\,t\,\sin\left(\frac{2\Gamma t}{\pi^2 d^2}\right)\right]\mathbf{x}\cdot(\mathbf{n}\,\mathbf{n})\cdot\mathbf{x}\right.$$
$$\left.+\left(\frac{2\Gamma^4 R_0}{\pi^3 d^4}-\frac{\Gamma^3 u_0}{\pi^2 d^2}\right)\cos\left(\frac{2\Gamma t}{\pi^2 d^2}\right)(\mathbf{x}\cdot\mathbf{x})\right\}$$

- Möhring's analogy:

$$p'_{\mathrm{M}}(\mathbf{x},t) = \frac{\rho_0}{4c_0^2|\mathbf{x}|^3}\left[\left(\frac{3\Gamma^3 u_0}{\pi^2 d^2}-\frac{4\Gamma^4 R_0}{\pi^3 d^4}\right)\cos\left(\frac{2\Gamma t}{\pi^2 d^2}\right)\right.$$
$$\left.\text{secular term}\quad -\frac{2\Gamma^4 u_0}{\pi^3 d^4}\,t\,\sin\left(\frac{2\Gamma t}{\pi^2 d^2}\right)\right]\mathbf{x}\cdot\left(\mathbf{n}\,\mathbf{n}-\frac{\mathbf{I}}{3}\right)\cdot\mathbf{x}$$

- Conclusion: failure of both Powell's and Möhring's analogies when applied to a flow model that does not respect the conservation of momentum and kinetic energy

Möhring's solution (trick ?): reinforcement of physical assumptions

$$p'_{\mathrm{M}}(\mathbf{x},t) = \frac{\rho_0}{4c_0^2|\mathbf{x}|^3}\frac{\mathrm{d}^3 Q}{\mathrm{d}t^3}\,\mathbf{x}\cdot\left(\mathbf{n}\,\mathbf{n}-\frac{\mathbf{I}}{3}\right)\cdot\mathbf{x}$$

- Using Lamb (1932) identities:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{T}{2\pi\rho_0} + 3\Gamma\left(R_{\mathrm{L}}V_{\mathrm{L}}Z_{\mathrm{L}} + R_{\mathrm{T}}V_{\mathrm{T}}Z_{\mathrm{T}}\right)$$

- Imposing further conservation of momentum:

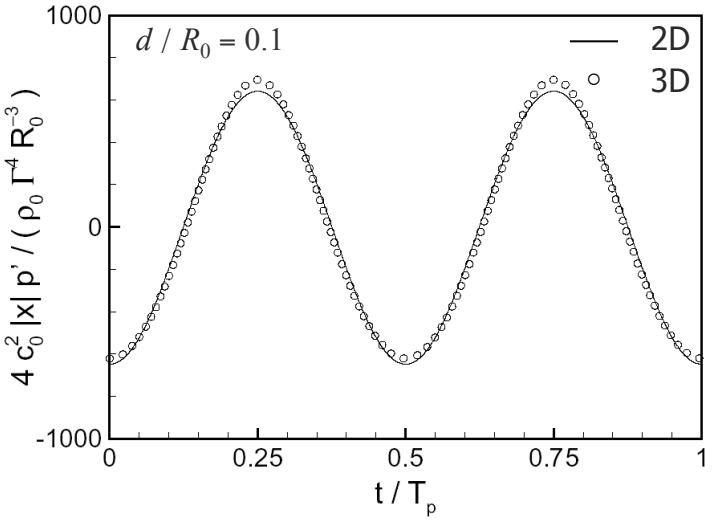
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(R_{\mathrm{L}}^2 + R_{\mathrm{T}}^2\right) = 0 \qquad R_{\mathrm{L}}V_{\mathrm{L}} = -R_{\mathrm{T}}V_{\mathrm{T}}$$

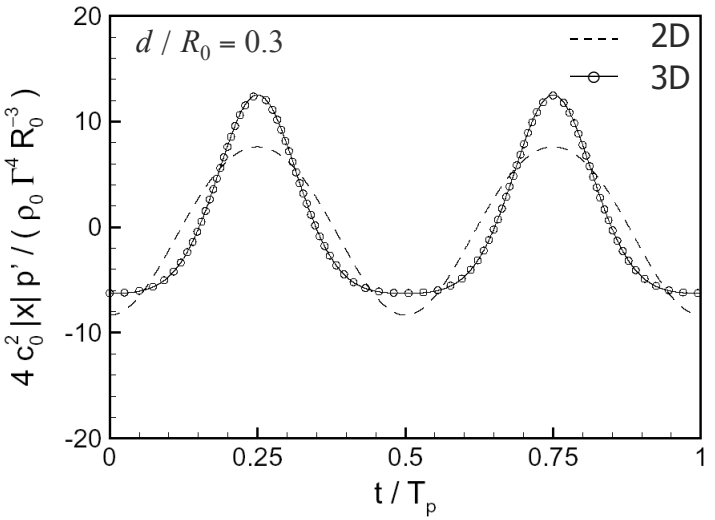
- We obtain:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{T}{2\pi\rho_0} + 3\Gamma R_{\mathrm{L}}V_{\mathrm{L}}\left(Z_{\mathrm{L}} - Z_{\mathrm{T}}\right)$$

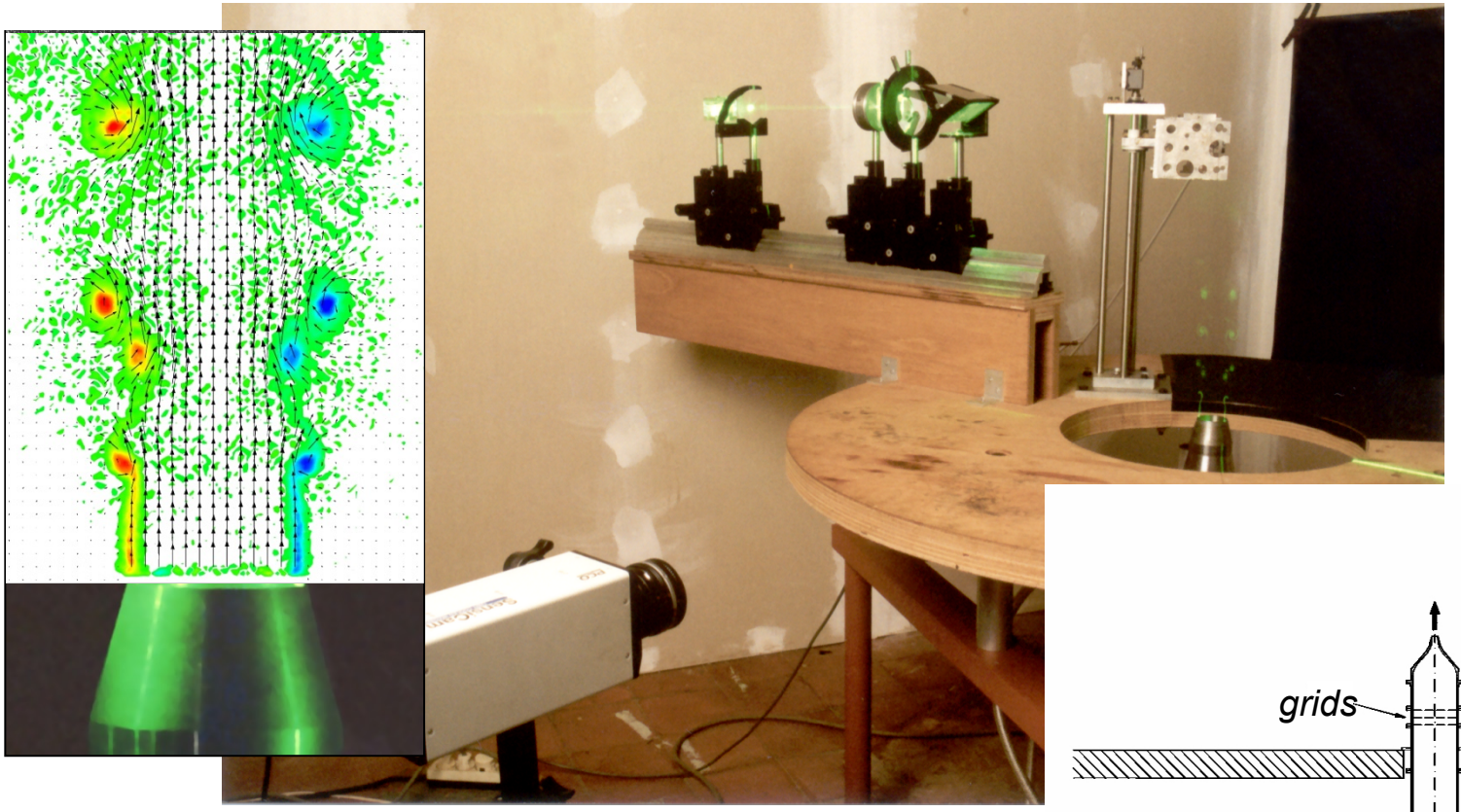
- Imposing further conservation of kinetic energy:

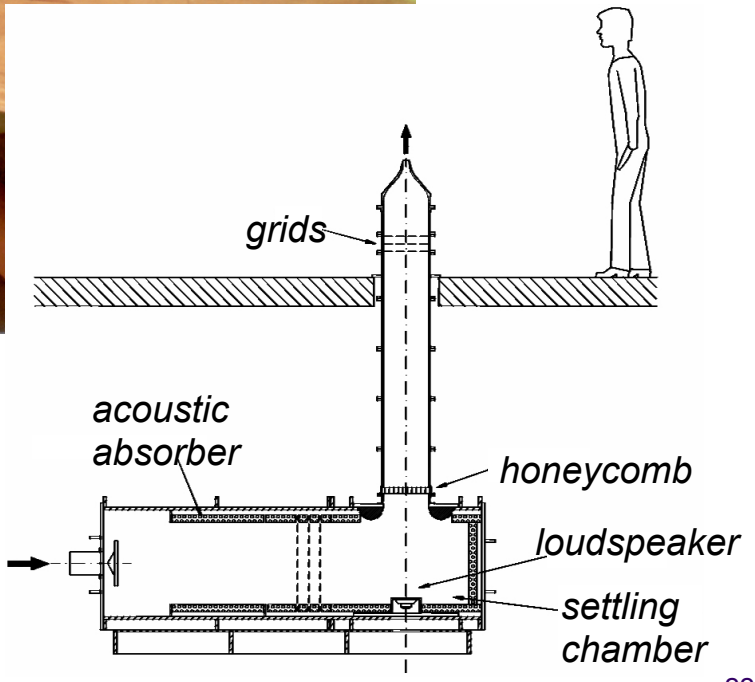
$$p'(\mathbf{x},t) = -\frac{3}{4}\frac{\rho_0\Gamma^4 R_0}{\pi^3 d^4 c_0^2|\mathbf{x}|^3}\cos\left(\frac{2\Gamma t}{\pi d^2}\right)\mathbf{x}\cdot\left(\mathbf{n}\mathbf{n}-\frac{\mathbf{I}}{3}\right)\cdot\mathbf{x}$$





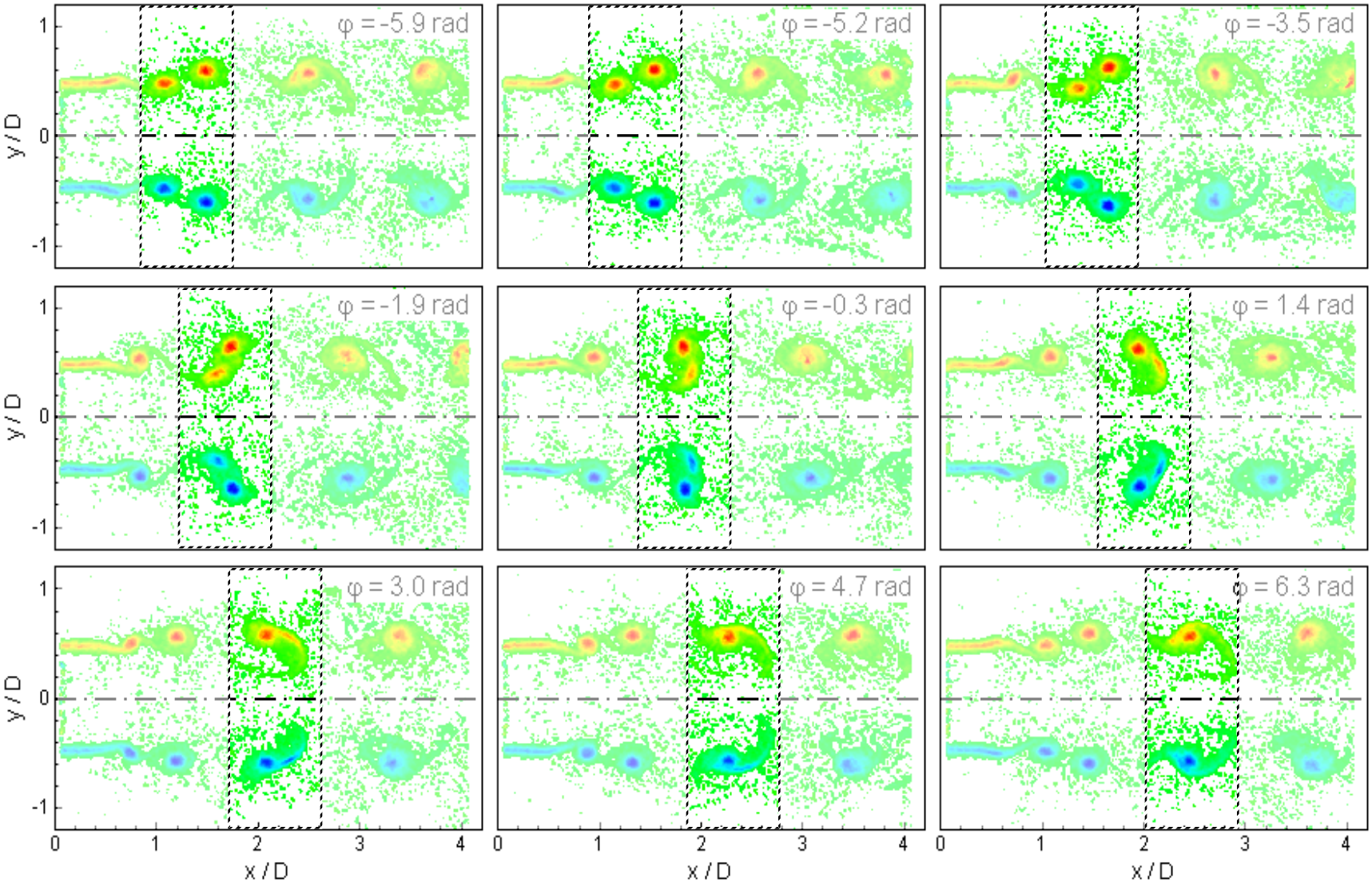
Experimental facility





- Two flow conditions:
  - U_0 = 5 m/s (Re = 14,000 → stable laminar vortices but negative decibels…)
  - U_0 = 34 m/s (Re = 93,000 → more turbulent vortices but measurable SPL)

Application to PIV data



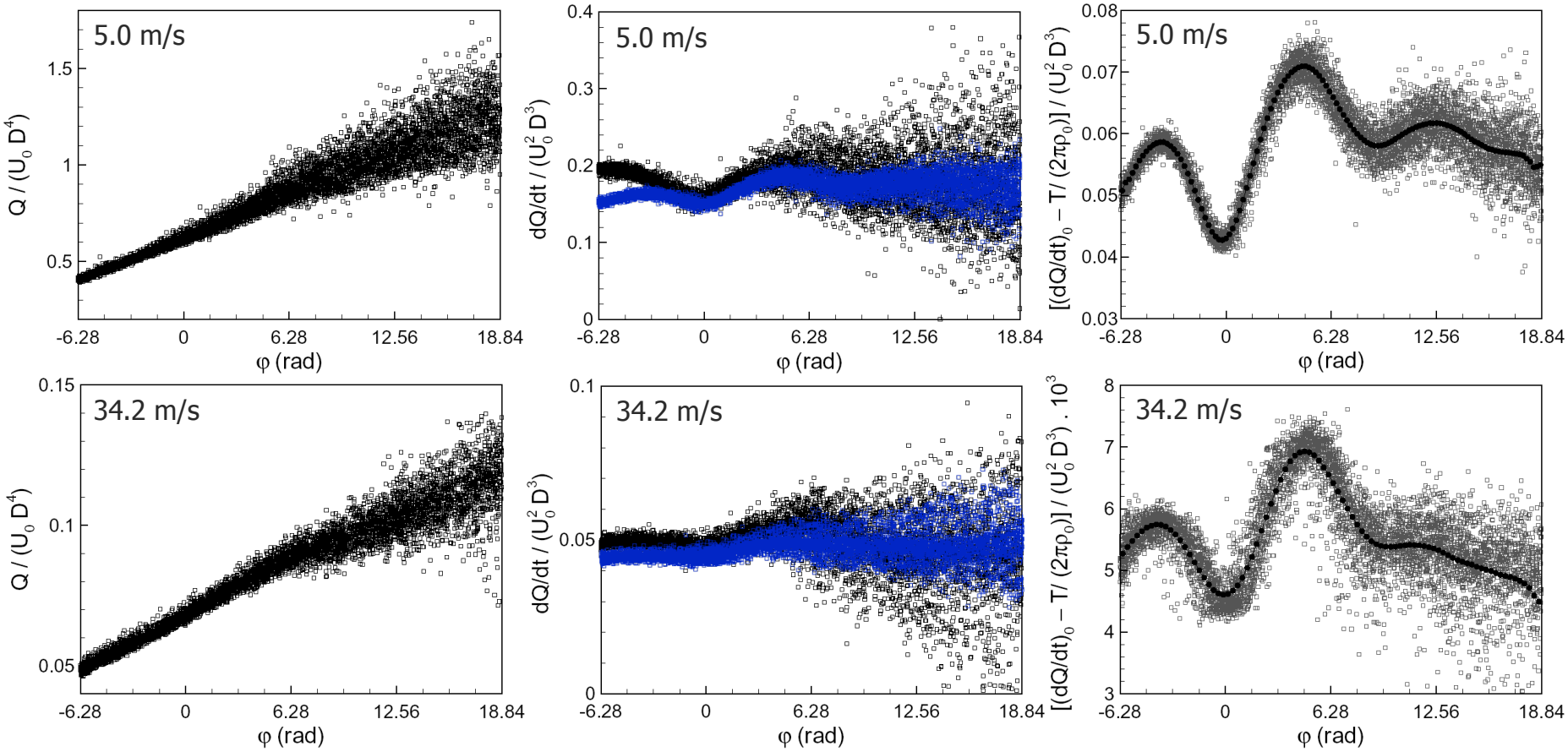
PIV results: acoustical source terms

$$\iint_S \omega r^2 z \, \mathrm{d}r\mathrm{d}z$$

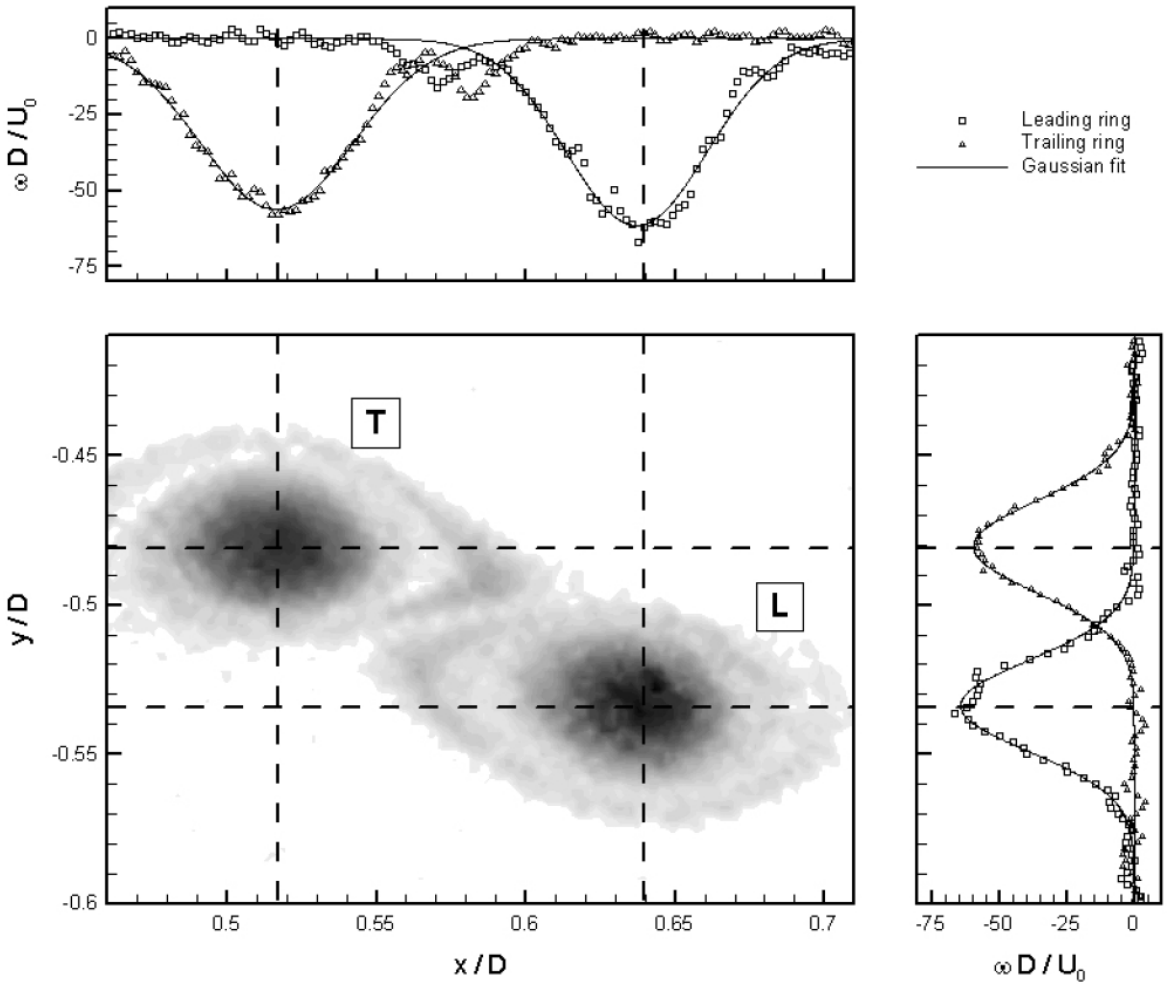
$$\frac{T}{2\pi\rho_0} + 3\iint_S \omega v r z \, \mathrm{d}r\mathrm{d}z$$

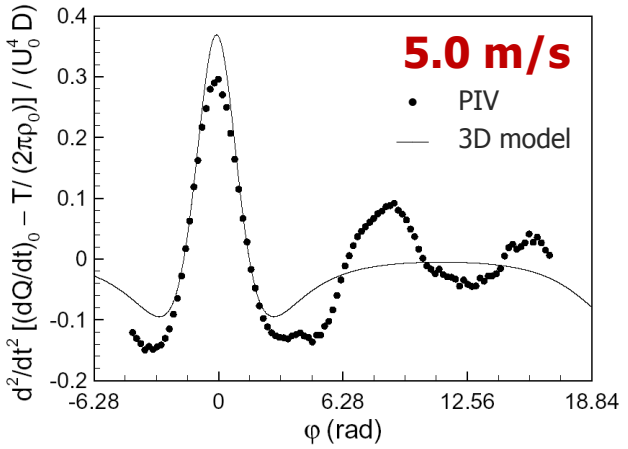
$$\boxed{\frac{T}{2\pi\rho_0} + 3\iint_S \omega v r \, (z - z_0) \, \mathrm{d}r\mathrm{d}z} \qquad z_0 = \frac{\iint_S \omega r^2 z \, \mathrm{d}r\mathrm{d}z}{\iint_S \omega r^2 \, \mathrm{d}r\mathrm{d}z}$$

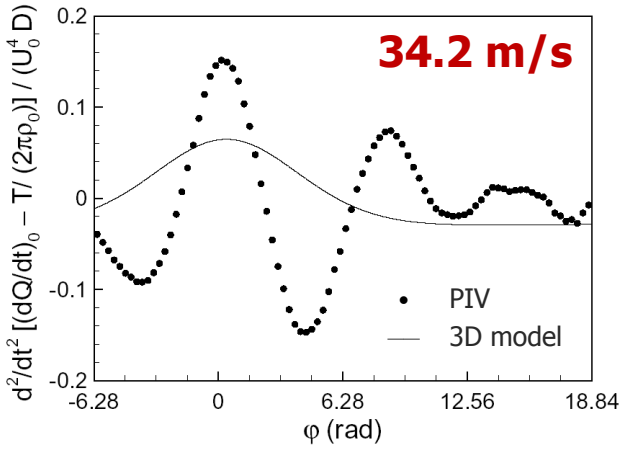
$$3\iint_S \omega v r \, (z - z_0) \, \mathrm{d}r\mathrm{d}z$$



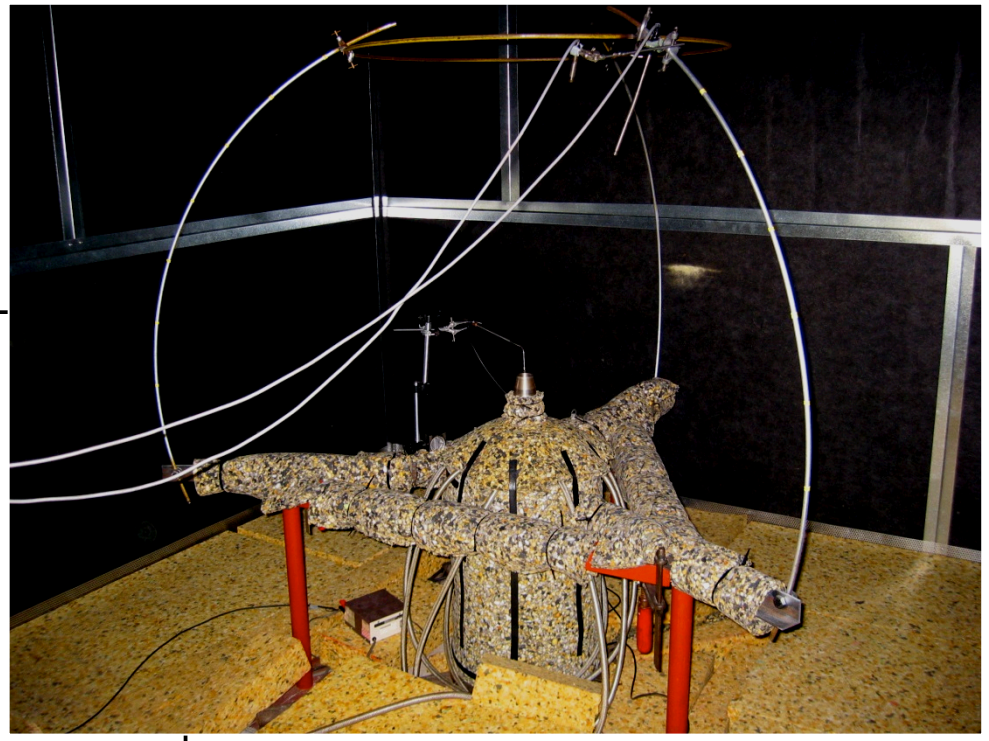
Acoustic predictions:
PIV vs. tuned analytical models

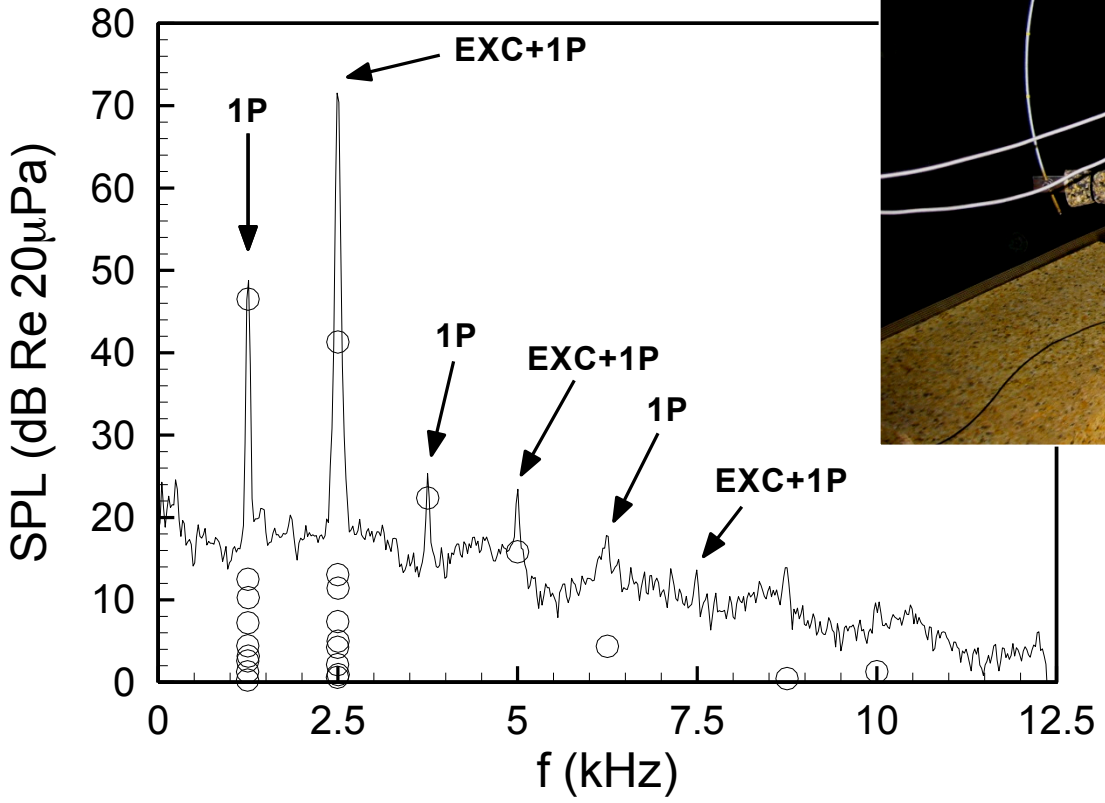




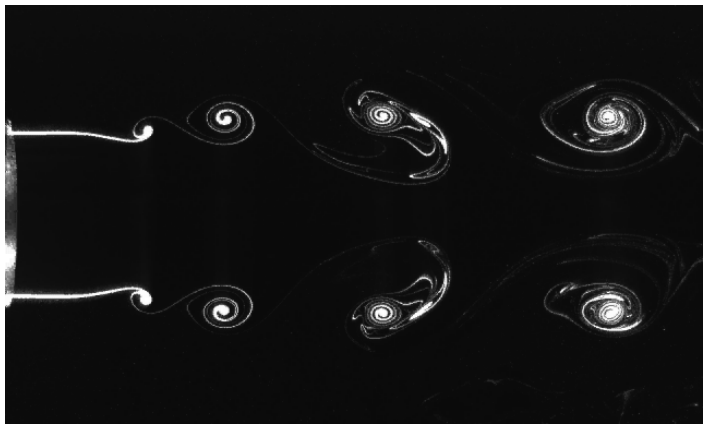


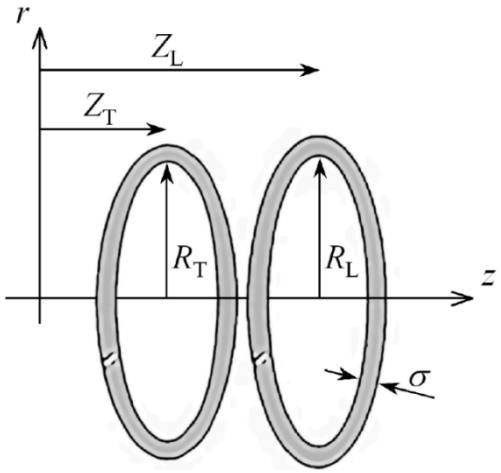
Acoustic predictions:
PIV vs. acoustic measurements





First question: why does a 2D model of periodic vortex pairing give linearly increasing pressure fluctuations ?





$$
\begin{aligned}
Z_L(t) &= u_0 t + \frac{d}{2}\cos\left(\frac{\Gamma t}{\pi d^2}\right)\\
R_L(t) &= R_0 + \frac{d}{2}\sin\left(\frac{\Gamma t}{\pi d^2}\right)\\
Z_T(t) &= u_0 t - \frac{d}{2}\cos\left(\frac{\Gamma t}{\pi d^2}\right)\\
R_T(t) &= R_0 - \frac{d}{2}\sin\left(\frac{\Gamma t}{\pi d^2}\right)
\end{aligned}
$$

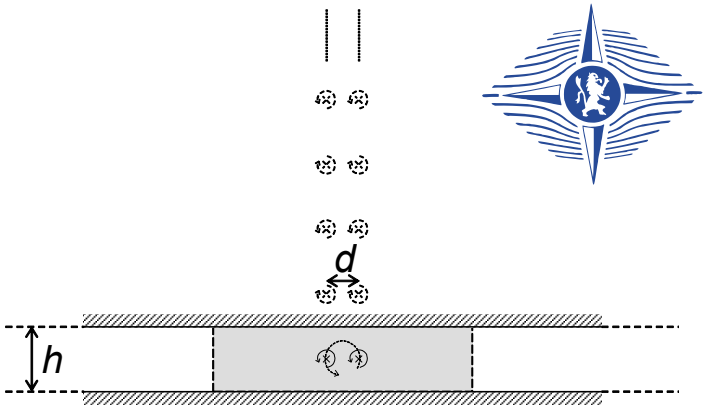
Answer: that's because I had let my acoustic analogy believe that my wrong flow model was… wrong!

$$
\begin{aligned}
p'(\mathbf{x},t) = \frac{\rho_0}{4c_0^2|\mathbf{x}|^3}\Bigg\{\Bigg[\Bigg(&-\frac{4\Gamma^4 R_0}{\pi^3 d^4}+\frac{3\Gamma^3 u_0}{\pi^2 d^2}\Bigg)\cos\left(\frac{2\Gamma t}{\pi^2 d^2}\right)\\
&-\frac{2\Gamma^4 u_0}{\pi^3 d^4}\,t\sin\left(\frac{2\Gamma t}{\pi^2 d^2}\right)\Bigg]\mathbf{x}\cdot(\mathbf{n}\,\mathbf{n})\cdot\mathbf{x}\\
&+\left(\frac{2\Gamma^4 R_0}{\pi^3 d^4}-\frac{\Gamma^3 u_0}{\pi^2 d^2}\right)\cos\left(\frac{2\Gamma t}{\pi^2 d^2}\right)(\mathbf{x}\cdot\mathbf{x})\Bigg\}
\end{aligned}
$$

???

$$
p'(\mathbf{x},t) = -\frac{3}{4}\frac{\rho_0\Gamma^4 R_0}{\pi^3 d^4 c_0^2|\mathbf{x}|^3}\cos\left(\frac{2\Gamma t}{\pi d^2}\right)\mathbf{x}\cdot\left(\mathbf{n}\mathbf{n}-\frac{\mathbf{I}}{3}\right)\cdot\mathbf{x}
$$

Second question: why does Curle's analogy <u>always</u> give me wrong results when I'm using an incompressible model for non-compact ducted flows ?

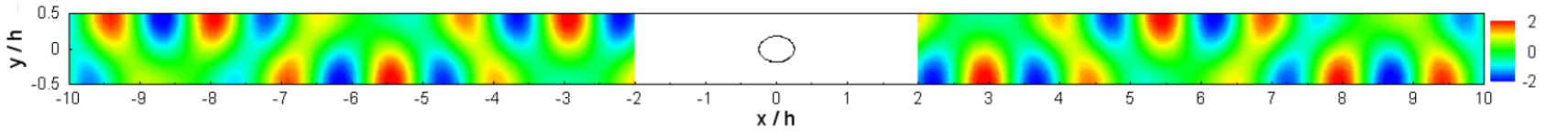


$$p_{\mathrm{a}}(\mathbf{x},\omega) = -\frac{\mathrm{i}}{2h}\sum_{n=0}^{\infty}\frac{\cos(\eta_n y)}{k_n C_n}$$

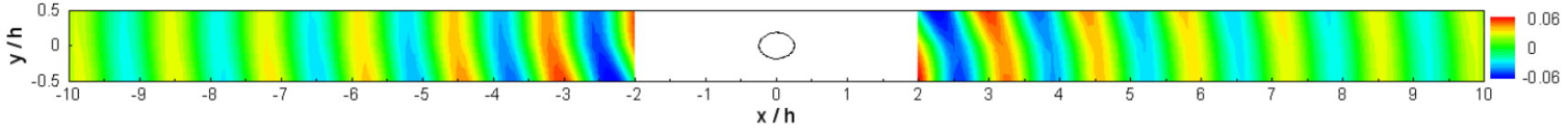
Exact solution (based on tailored Green's function):

$$\Big\{k_n^2 \iint_{S_0} \cos(\eta_n y_0)\,\rho_0 u^2\,\mathrm{e}^{\mp \mathrm{i}k_n(x-x_0)}\,\mathrm{d}x_0\,\mathrm{d}y_0$$
$$+\,\eta_n^2 \iint_{S_0} \cos(\eta_n y_0)\,\rho_0 v^2\,\mathrm{e}^{\mp \mathrm{i}k_n(x-x_0)}\,\mathrm{d}x_0\,\mathrm{d}y_0$$
$$\pm\,\mathrm{i}k_n\eta_n \iint_{S_0} \sin(\eta_n y_0)\,\rho_0 uv\,\mathrm{e}^{\mp \mathrm{i}k_n(x-x_0)}\,\mathrm{d}x_0\,\mathrm{d}y_0\Big\}$$

Exact:



Curle:



Curle's analogy: fixed rigid bodies

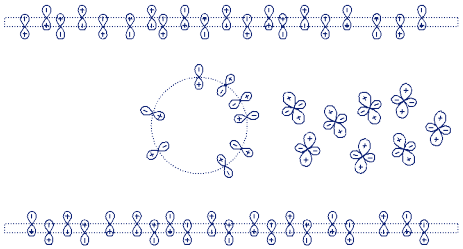
- Lighthill's aeroacoustical analogy: $\dfrac{\partial^2 \rho'}{\partial t^2} - c_0^2 \dfrac{\partial^2 \rho'}{\partial x_i^2} = \dfrac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$

- Integral solution using Green's function

$$\rho'(\mathbf{x},t) = \int_{-\infty}^{t} \iiint_V \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} G \, \mathrm{d}^3\mathbf{y} \mathrm{d}\tau \qquad \textit{incident field}$$

$$- c_0^2 \int_{-\infty}^{t} \iint_{\partial V} \left(\rho' \frac{\partial G}{\partial y_i} - G \frac{\partial \rho'}{\partial y_i} \right) n_i \, \mathrm{d}^2\mathbf{y} \, \mathrm{d}\tau$$

scattered field



- Partial integration of source integral

$$\int_{-\infty}^{t} \iiint_V \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} G \, \mathrm{d}^3\mathbf{y} \mathrm{d}\tau = \int_{-\infty}^{t} \iiint_V T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} \, \mathrm{d}^3\mathbf{y} \, \mathrm{d}\tau$$

$$+ \int_{-\infty}^{t} \iint_{\partial V} \left\{ \left(- \frac{\partial \rho v_i}{\partial \tau} - \underline{c_0^2 \frac{\partial \rho'}{\partial y_i}} \right) G - \left(\rho v_i v_j + (p' - \underline{c_0^2 \rho'}) \, \delta_{ij} + \sigma_{ij} \right) \frac{\partial G}{\partial y_j} \right\} n_i \, \mathrm{d}^2\mathbf{y} \, \mathrm{d}\tau$$

- Curle's analogy: uses free field Green's function $G_0(t,\mathbf{x}|\tau,\mathbf{y}) = \dfrac{\delta(t-\tau-|\mathbf{x}-\mathbf{y}|/c_0)}{4\pi c_0^2 |\mathbf{x}-\mathbf{y}|}$

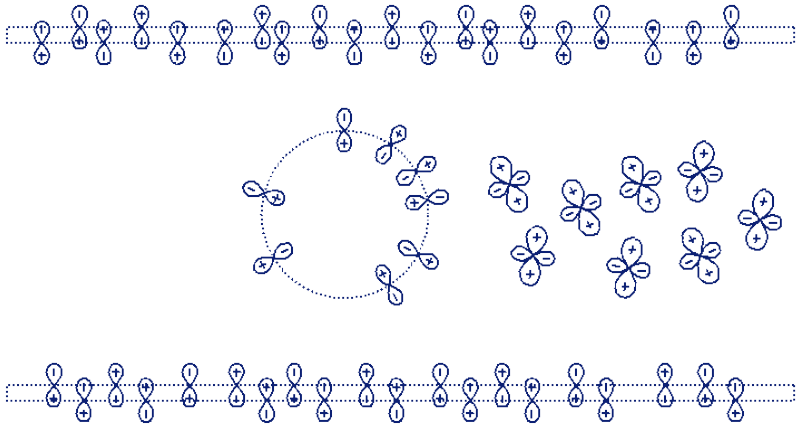
$$\Rightarrow \rho'(\mathbf{x},t) = \frac{\partial^2}{\partial x_i \partial x_j} \iiint_V \left[\frac{T_{ij}}{4\pi c_0^2 |\mathbf{x}-\mathbf{y}|} \right] \mathrm{d}^3\mathbf{y} - \frac{\partial}{\partial x_i} \iint_{\partial V} \left[\frac{p' n_i}{4\pi c_0^2 |\mathbf{x}-\mathbf{y}|} \right] \mathrm{d}^2\mathbf{y}$$

Quadrupole, $W \propto M^8$ in free field

Dipole, $W \propto M^6$ in free field

The black magic behind Curle's acoustic dipoles

- No assumption made in Curle's analogy, only a reformulation of Navier-Stokes equations solved using Green's function
- But hard-wall boundary condition has disappeared, replaced by equivalent dipoles distributed over solid surfaces



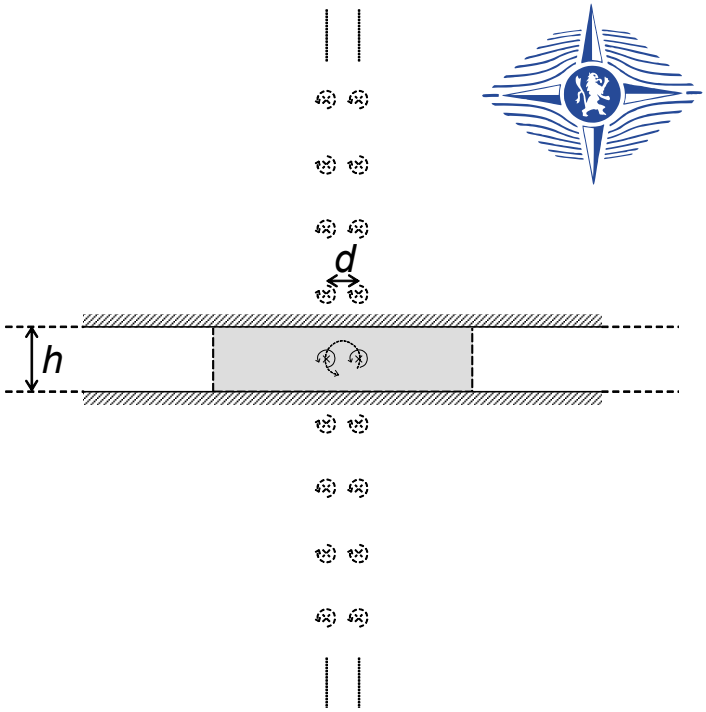
→ The dipoles must represent the non-penetration boundary condition, for

- the hydrodynamic problem: reaction force exerted by the walls subjected to wall-normal flow momentum, and
- the acoustic problem: scattering of incident acoustic waves.

- **But what if my dipoles are computed from an incompressible flow model ?**

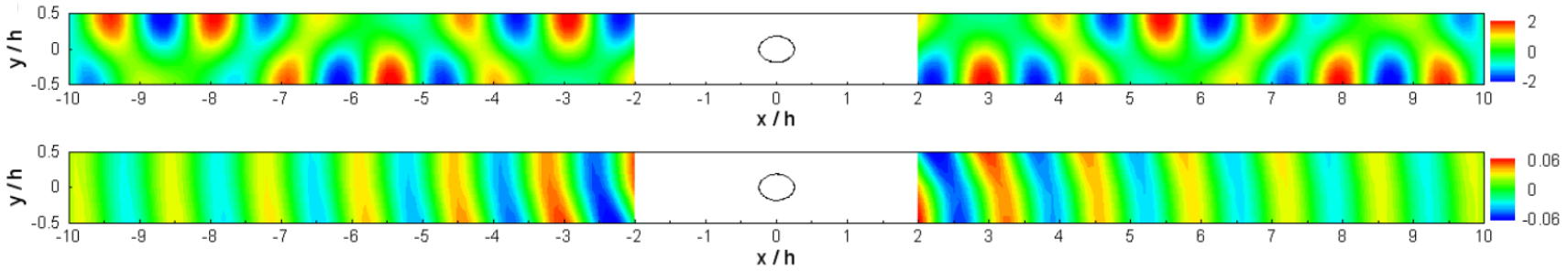
Second question: why does Curle's analogy <u>always</u> give me wrong results when I'm using an incompressible model for non-compact ducted flows ?

Answer: that's because the dipoles lack the compressible component corresponding to acoustic scattering, leading to acoustic leakage effects.



Exact:

Curle:



Does it imply incompressible models cannot be used for such cases ?

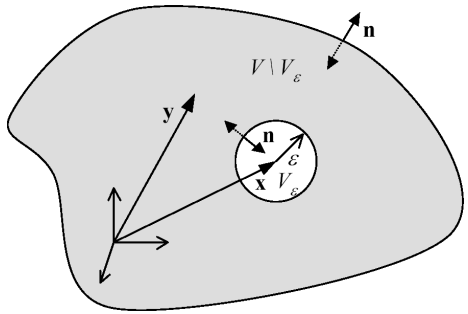
No, but the distinction between hydrodynamic and acoustic effects must be introduced explicitly.

Boundary Integral discretization: bringing the listener inside the source region

- Direct Boundary Element Method (DBEM)
  - Derivation essentially similar to Curle's analogy
    - Resolution of the Helmholtz equation $\nabla^2 \hat{p}_a + k^2 \hat{p}_a = q_L$

      with the source term $\hat{q}_L = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(-\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \right) e^{-i\omega t} \, \mathrm{d}t \equiv -\frac{\partial^2 \hat{T}_{ij}}{\partial x_i \partial x_j}$

      using free field Green's function $G = \frac{\mathrm{e}^{-ikr}}{4\pi r}$, $r = |\mathbf{x} - \mathbf{y}|$ $\quad \nabla^2 G + k^2 G = -\delta(\mathbf{x} - \mathbf{y})$
  - Collocation method: the integral solution is evaluated over the acoustic mesh
    - Exclude Green's kernel singularity!



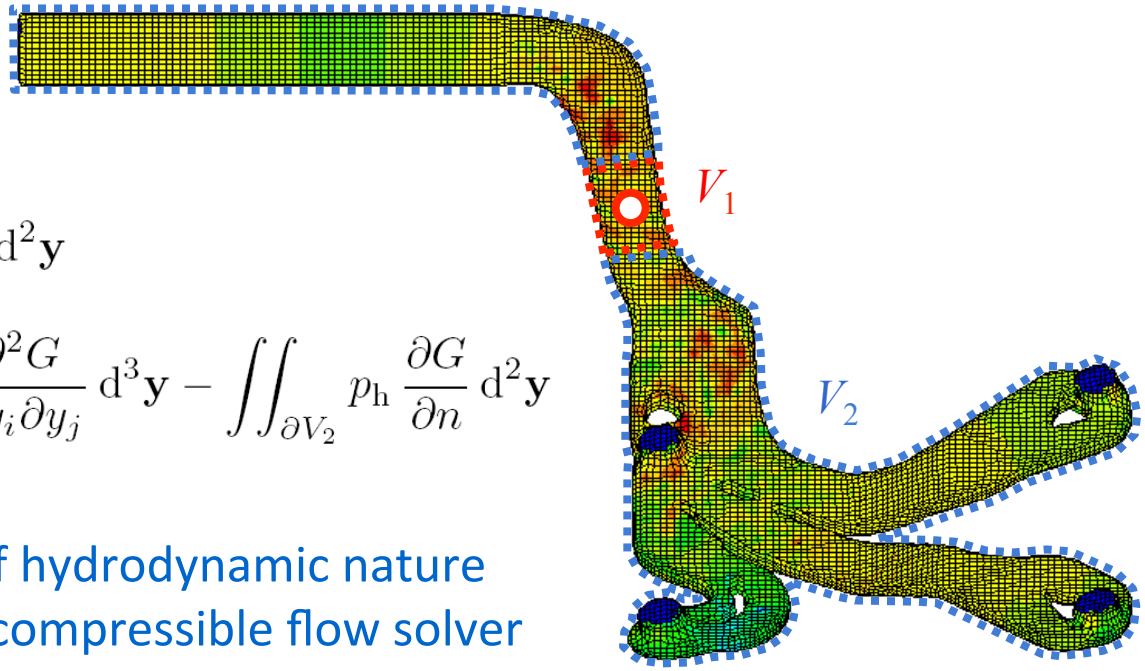
$$\int_{V \setminus V_\varepsilon} \left(\nabla^2 p_a \; G - p_a \; \nabla^2 G \right) \; \mathrm{d}^3\mathbf{y} = \int_{V \setminus V_\varepsilon} q_L \, G \;\; \mathrm{d}^3\mathbf{y} + \int_{V \setminus V_\varepsilon} p_a \; \delta(\mathbf{x} - \mathbf{y}) \;\; \mathrm{d}^3\mathbf{y}$$

$$\Rightarrow \quad \boxed{C(\mathbf{x})\, p_{\mathrm{L}}(\mathbf{x}) = \iiint_V T_{ij} \, \frac{\partial^2 G}{\partial y_i \partial y_j} \, \mathrm{d}^3\mathbf{y} - \iint_{\partial V} p_{\mathrm{L}} \, \frac{\partial G}{\partial n} \, \mathrm{d}^2\mathbf{y}}$$

Contributions of compact and non-compact regions

- We decompose the pressure into acoustic and hydrodynamic: $p_{\mathrm{L}} = p_{\mathrm{h}} + p_{\mathrm{a}}$

$$C(\mathbf{x})\, p_{\mathrm{a}}(\mathbf{x}) = -\iint_{\partial V} p_{\mathrm{a}} \frac{\partial G}{\partial n}\, \mathrm{d}^2\mathbf{y} + \iiint_{V_2} T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j}\, \mathrm{d}^3\mathbf{y} - \iint_{\partial V_2} p_{\mathrm{h}} \frac{\partial G}{\partial n}\, \mathrm{d}^2\mathbf{y}$$

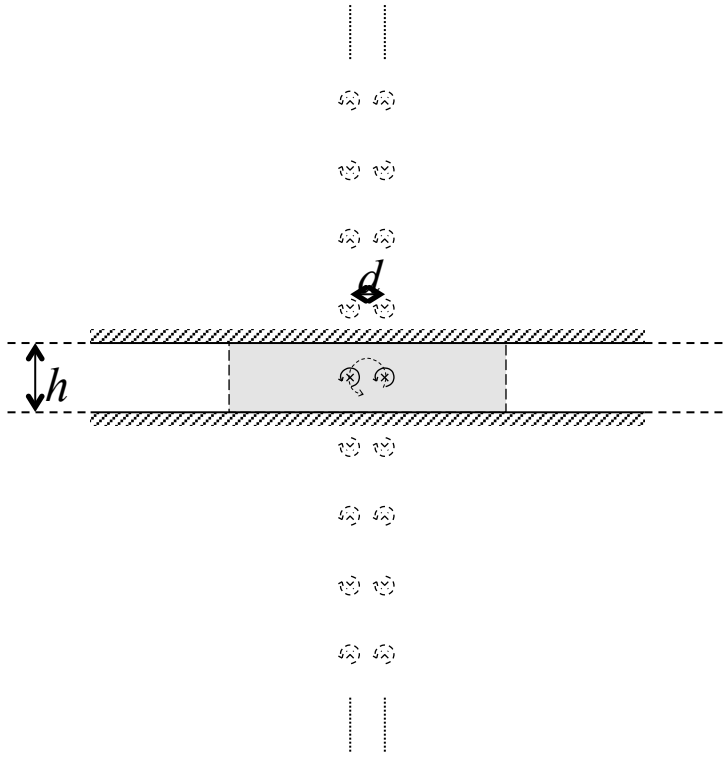


- The near-field effects are of hydrodynamic nature and taken care of by the incompressible flow solver
- The far-field effects are of compressible nature and obtained through the BEM solver

Validation: spinning vortex pair in straight duct

- Purpose: perform unambiguous validation of the BEM / Curle approach
  - Flow should be amenable to (nearly) exact modelling
  - Geometry should allow an exact evaluation of the scattering (tailored Green's fct)
  - Incompressible flow model

- Leapfrogging of 2 rectilinear vortex filaments in an infinite 2D duct
  - Flow kinematics: based on the complex potentials of the system of 2 vortices, and of the infinite series of image vortices
  - Non-penetration, slip condition at both walls (streamlines)



Flow model: source fields and vortex desingularization

- Volume source field:
  1. Compute the vortex kinematics by time marching the equations

$$u_m = -\frac{\Gamma}{4h}\left\{ \frac{\sin\left[\pi\left(y_m - y_n\right)/h\right]}{\cosh\left[\pi\left(x_m - x_n\right)/h\right] - \cos\left[\pi\left(y_m - y_n\right)/h\right]} + \frac{\sin\left[\pi\left(y_m + y_n\right)/h\right]}{\cosh\left[\pi\left(x_m - x_n\right)/h\right] + \cos\left[\pi\left(y_m + y_n\right)/h\right]} + \frac{\sin\left(2\pi y_m/h\right)}{1 + \cos\left(2\pi y_m/h\right)} \right\}$$

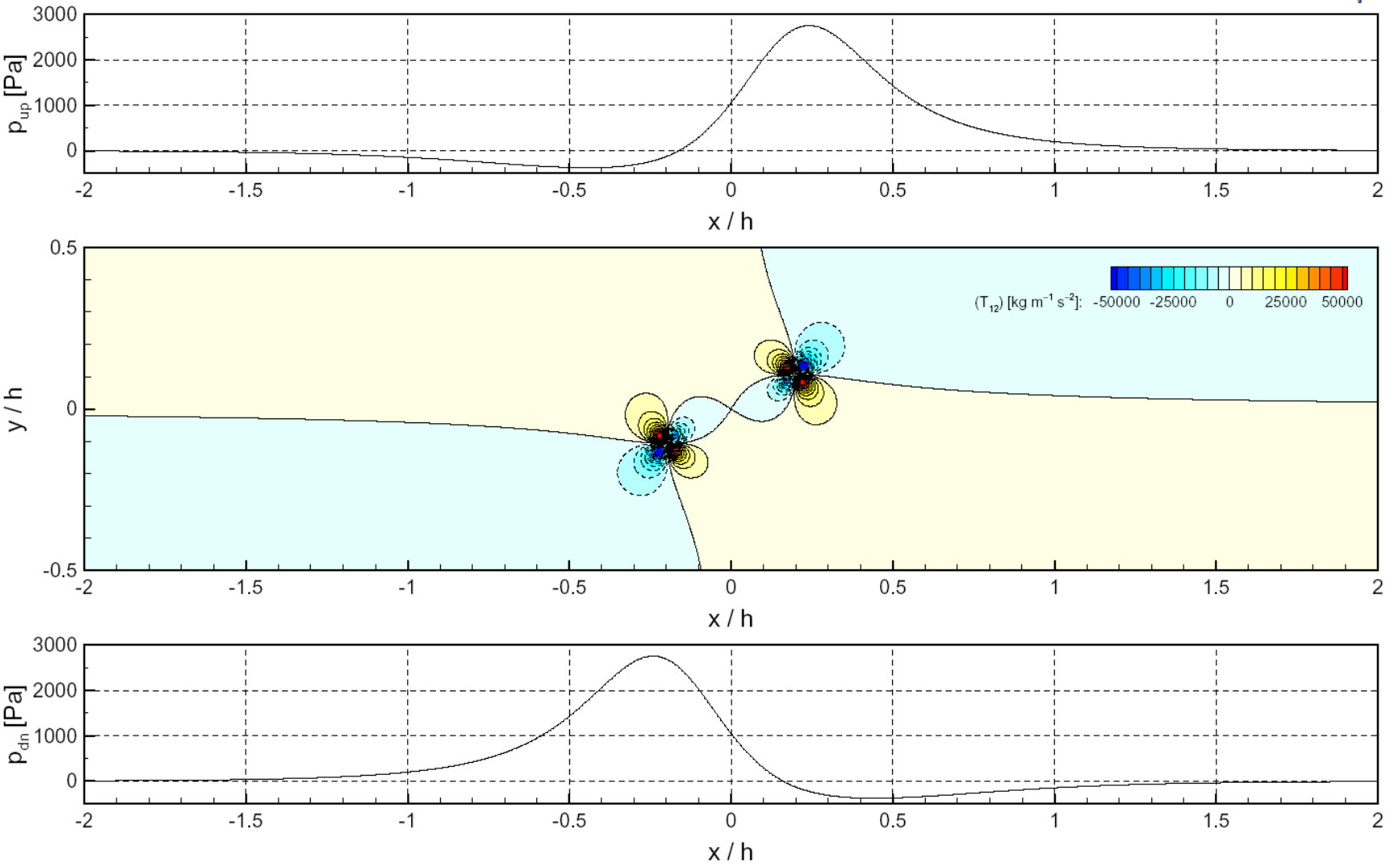
$$v_m = -\frac{\Gamma}{4h}\left\{ \frac{-\sinh\left[\pi\left(x_m - x_n\right)/h\right]}{\cosh\left[\pi\left(x_m - x_n\right)/h\right] - \cos\left[\pi\left(y_m - y_n\right)/h\right]} + \frac{\sinh\left[\pi\left(x_m - x_n\right)/h\right]}{\cosh\left[\pi\left(x_m - x_n\right)/h\right] + \cos\left[\pi\left(y_m + y_n\right)/h\right]} \right\}$$

  2. Compute the induced velocity field using the desingularized kernel

$$v_\theta(r) = \frac{\Gamma}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{2\sigma^2}\right)\right]$$

- Wall pressure: integrated from unsteady Bernoulli's eq.: $p_w = -\rho\left(\frac{\partial \Phi_w}{\partial t} + \frac{u_w^2}{2}\right)$

Source fields: T_{xy} and wall pressure



Reference solution used for validation

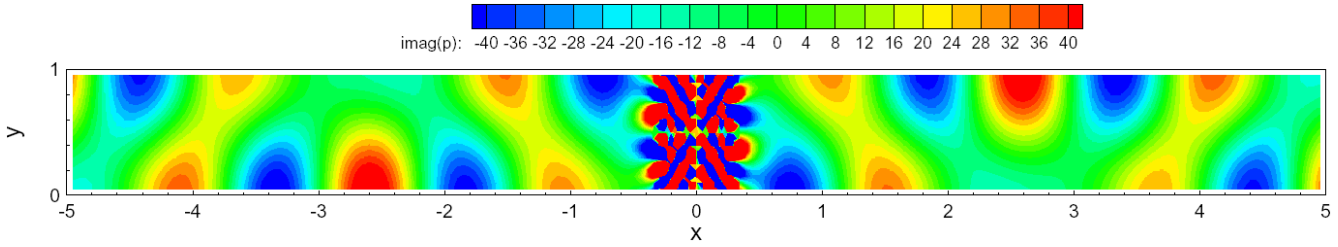
- Reference solution: based on tailored Green's function

$$G_1 = \frac{\mathrm{i}}{2h} \sum_{n=0}^{\infty} \frac{1}{C_n k_n} \cos(\eta_n y_0) \cos(\eta_n y) \, \mathrm{e}^{\mp \mathrm{i} k_n (x - x_0)} \qquad C_n = \begin{cases} 1 & \text{if} \quad n = 0 \\ 1/2 & \text{if} \quad n \neq 0 \end{cases}$$

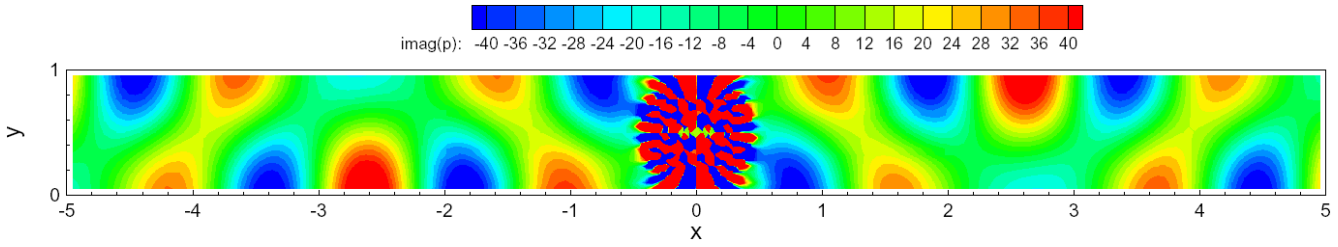
$$\Rightarrow \quad p_{\mathrm{a}}(\mathbf{x}, \omega) = -\frac{\mathrm{i}}{2h} \sum_{n=0}^{\infty} \frac{\cos(\eta_n y)}{k_n C_n} \left\{ k_n^2 \iint_{S_0} \cos(\eta_n y_0) \, \rho_0 u^2 \, \mathrm{e}^{\mp \mathrm{i} k_n (x - x_0)} \, \mathrm{d}x_0 \, \mathrm{d}y_0 \right.$$
$$+ \eta_n^2 \iint_{S_0} \cos(\eta_n y_0) \, \rho_0 v^2 \, \mathrm{e}^{\mp \mathrm{i} k_n (x - x_0)} \, \mathrm{d}x_0 \, \mathrm{d}y_0$$
$$\left. \pm \mathrm{i} k_n \eta_n \iint_{S_0} \sin(\eta_n y_0) \, \rho_0 u v \, \mathrm{e}^{\mp \mathrm{i} k_n (x - x_0)} \, \mathrm{d}x_0 \, \mathrm{d}y_0 \right\}$$

Validation of the Curle/DBEM method for *kh* = 4.8

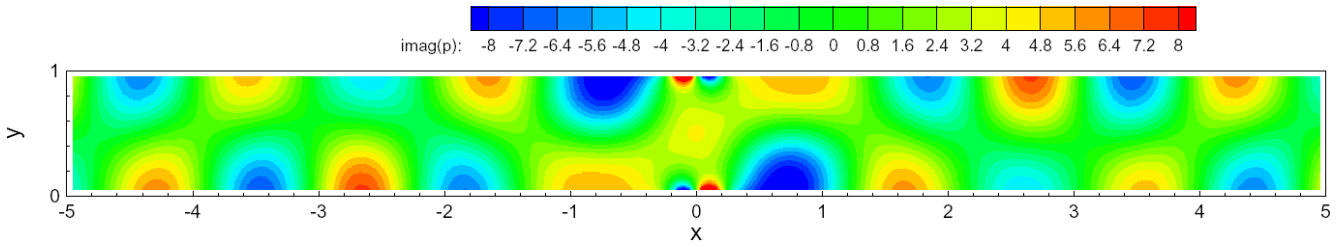
Reference solution



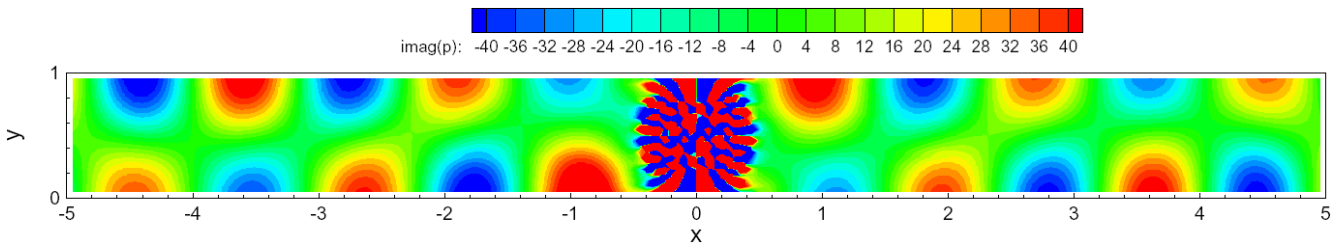
Curle/DBEM



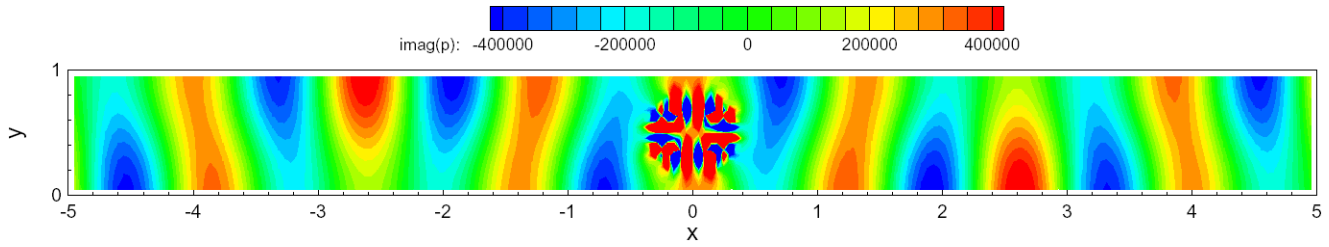
Dipolar contribution



Quadrupolar contribution



Lighthill



Summary

- Aeroacoustical analogies allow extracting a maximum of acoustical information from a given description of the flow field
- In absence of flow-acoustic coupling, an incompressible flow model permits to obtain a reasonable acoustic prediction

- Numerical robustness issues have been reviewed:
  - In cases of non-compact source regions, the incompressible solution must be complemented by an acoustic correction to account for scattering effects
  - The prediction of free jet quadrupolar noise requires a flow model free of spurious monopolar or dipolar sources , which can be otherwise explicitly discarded using an appropriate source formulation

- Without approximations, the analogy is useless!

Thanks for your attention!



Questions ?