Problem Set on O Δ E

1. The trapezoidal method $u_{n+1} = u_n + \frac{1}{2}h(u'_{n+1} + u'_n)$ is used to solve the ODE $u' = \lambda u + a$ numerically.

   (a) What is the resulting OΔE ?

   (b) What is its exact numerical solution ?

   (c) How does the exact steady state solution of the OΔE compare with the exact steady state solution of the ODE (Hint:The exact SS solution is $u(t \to \infty) = -\frac{a}{\lambda}$)?

2. Consider the ODE

$$\mathbf{u}' = \frac{d\mathbf{u}}{dt} = [A]\mathbf{u} + \mathbf{f}$$

   with

$$[A] = \begin{bmatrix} -10 & -0.1 & -0.1 \\ 1 & -1 & 1 \\ 10 & 1 & -1 \end{bmatrix} \quad , \quad \mathbf{f} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

   (a) Find the eigenvalues of $[A]$ using Matlab. What is the long time Steady State (SS) solution $\mathbf{u}$? How would the ODE solution behave in time? (Hint: Remember the $e^{\lambda t}$ form of ODE solutions.)

   (b) Write a Matlab code to integrate from the initial condition $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ from time $t = 0$ for the three time advance schemes ($h = \Delta t$)

      i. $u_{n+1} = u_n + h(u')_n$ the Euler Explicit Scheme

      ii. $u_{n+1} = u_n + h(u')_{n+1}$ the Euler Implicit Scheme

      iii. $u_{n+\frac{1}{2}} = u_n + h(u')_n \quad ; \quad u_{n+1} = u_n + \frac{1}{2}h\left((u')_{n+\frac{1}{2}} + (u')_n\right)$ the Predictor-Corrector Scheme

      In all three cases use $h = 0.1$ for 1000 time steps, $h = 0.2$ for 500 time steps, $h = 0.4$ for 250 time steps and $h = 1.0$ for 100 time steps. Compare the computed SS solution with the exact SS solution.

   (c) Could you have predicted the behavior of the previous problem? In class we developed the $\sigma - \lambda$ relations for these methods.

      i. For the Euler Explicit Scheme: $\sigma = (1 + h\lambda)$.

      ii. For the Euler Implicit Scheme: $\sigma = 1/(1 - h\lambda)$.

      iii. For the Predictor-Corrector Scheme: $\sigma = (1 + h\lambda + \frac{1}{2}(h\lambda)^2)$.

      The stability condition is $|\sigma| \leq 1.0$. For the Euler Explicit scheme what is the predicted stability limit on h and is it confirmed by your Matlab code? (Hint: Try running just below and above the limit, also use the eigenvalues from 2(a) in the stability check).

3. The "backward differentiation" scheme is given by

$$u_{n+1} = \frac{1}{3}\left[4u_n - u_{n-1} + 2hu'_{n+1}\right]$$

   (a) Write the OΔE for the representative equation $u' = \lambda u + ae^{\mu t}$. Indentify the polynomials $P(E)$ and $Q(E)$.

   (b) Derive the $\lambda - \sigma$ relation. Are there multiple roots and if so identify the spurious ones.

   (c) Find er_λ.

   (d) Find the first two nonvanishing terms in a Taylor series expansion of all the spurious roots.

4. Consider the time march scheme given by

$$u_{n+1} = u_{n-1} + \frac{2h}{3}(u'_{n+1} + u'_n + u'_{n-1})$$

   (a) Write the OΔE for the representative equation $u' = \lambda u + a$. Indentify the polynomials $P(E)$ and $Q(E)$.

   (b) Derive the $\lambda - \sigma$ relation.

   (c) Find er_λ.

5. Consider the predictor–corrector combination

$$\begin{aligned} \tilde{u}_{n+1} &= u_n + hu'_n \\ u_{n+1} &= \alpha_1 u_n + \alpha_2 \tilde{u}_{n+1} + \beta h \tilde{u}'_{n+1} \end{aligned}$$

   (a) Find the values of α_1, α_2 and β that minimize er_λ.

   (b) Using this method, find the exact numerical solution to $u' = \lambda\, u + ae^{\mu t}$. Do not expand to find er_μ

6. Find the value of er_ω (assuming the ODE $u' = i\omega u$) for:

   (a) The 1st order Runge–Kutta method.

   (b) The 2nd order Runge–Kutta method.

   (c) Do the numerical solutions, found by using the above methods, lead or lag the exact solution to $u' = i\omega u$? A wave lags if it takes longer for a crest to materialize.