Answers on Modified Wave Numbers

1. Find the expression for the modified wave number in the following centered difference approximations to $(\delta_x u)_j$ in terms of Δx and k. This is done just as done in class where we let $u_j = e^{ikj\Delta x}$. (Cast the results in terms of $sin(k\Delta x)$ and $cos(k\Delta x)$).

   (a) $(\delta_x u)_j = (u_{j+1} - u_{j-1})/(2\Delta x)$
   We apply $u_j = e^{ikj\Delta x}$ to both sides and get

   $$\left(ik' e^{ikj\Delta x}\right) = e^{ikj\Delta x}\left(e^{+ik\Delta x} - e^{-ik\Delta x}\right)/(2\Delta x)$$

   which give us

   $$ik' = i\frac{sin(k\Delta x)}{\Delta x}$$

   (b) $(\delta_x u)_j = (-u_{j+2} + 8u_{j+1} - 8u_{j-1} + u_{j-2})/(12\Delta x)$
   We apply $u_j = e^{ikj\Delta x}$ to both sides and get

   $$\left(ik' e^{ikj\Delta x}\right) = e^{ikj\Delta x}\left(-e^{2ik\Delta x} + 8e^{+ik\Delta x} - 8e^{-ik\Delta x} + e^{-2ik\Delta x}\right)/(12\Delta x)$$

   which give us

   $$ik' = i\frac{\left(\frac{4}{3}sin(k\Delta x) - \frac{1}{6}sin(2k\Delta x)\right)}{\Delta x}$$

   (c) $\frac{1}{6}\left((\delta_x u)_{j+1} + 4(\delta_x u)_j + (\delta_x u)_{j-1}\right) = (u_{j+1} - u_{j-1})/(2\Delta x)$
   We apply $u_j = e^{ikj\Delta x}$ to both sides and get

   $$\frac{1}{6}e^{ikj\Delta x}\left(ik' e^{+ik\Delta x} + 4ik' + ik' e^{-ik\Delta x}\right) = e^{ikj\Delta x}\left(e^{+ik\Delta x} - e^{-ik\Delta x}\right)/(2\Delta x)$$

   which give us

   $$\frac{1}{3}\left(2 + cos(k\Delta x)\right)ik' = i\frac{sin(k\Delta x)}{\Delta x}$$

   or

   $$ik' = i\frac{3sin(k\Delta x)}{\Delta x(2 + cos(k\Delta x))}$$

2. Find the expression for the modified wave number in the following one sided difference approximations to $(\delta_x u)_j$ in terms of Δx and k. In this case there will be real and imaginary parts to the modified wave number. (Cast the results in terms of $sin(k\Delta x)$ and $cos(k\Delta x)$).

   (a) $(\delta_x u)_j = (u_j - u_{j-1})/\Delta x$ We apply $u_j = e^{ikj\Delta x}$ to both sides and get

   $$\left(ik' e^{ikj\Delta x}\right) = e^{ikj\Delta x}\left(1 - e^{-ik\Delta x}\right)/(\Delta x)$$

   which give us

   $$ik' = \frac{1 - cos(k\Delta x)}{\Delta x} + i\frac{sin(k\Delta x)}{\Delta x}$$

   (b) $(\delta_x u)_j = (3u_j - 4u_{j-1} + u_{j-2})/(2\Delta x)$
   We apply $u_j = e^{ikj\Delta x}$ to both sides and get

   $$\left(ik' e^{ikj\Delta x}\right) = e^{ikj\Delta x}\left(3 - 4e^{-ik\Delta x} + e^{-2ik\Delta x}\right)/(2\Delta x)$$

   which give us

   $$ik' = \frac{3 - 4cos(k\Delta x) + cos(2k\Delta x)}{2\Delta x} + i\frac{4sin(k\Delta x) - sin(2k\Delta x)}{2\Delta x}$$

(c) $2(\delta_x u)_j + (\delta_x u)_{j-1} = (u_{j+1} + 4u_j - 5u_{j-1})/(2\Delta x)$
We apply $u_j = e^{ikj\Delta x}$ to both sides and get

$$e^{ikj\Delta x}\left(2ik' + ik'e^{-ik\Delta x}\right) = e^{ikj\Delta x}\left(e^{+ik\Delta x} + 4 - 5e^{-ik\Delta x}\right)/(2\Delta x)$$

which give us

$$(2 + cos(k\Delta x) - isin(k\Delta x))\, ik' = \frac{2 - 2cos(k\Delta x) + 3isin(k\Delta x)}{\Delta x}$$

or

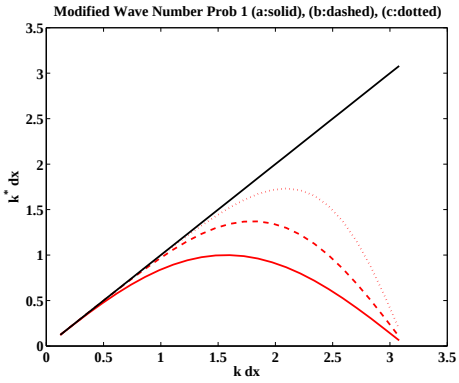
$$ik' = \frac{2 - 2cos(k\Delta x) + 3isin(k\Delta x)}{(2 + cos(k\Delta x) - isin(k\Delta x))\,\Delta x}$$

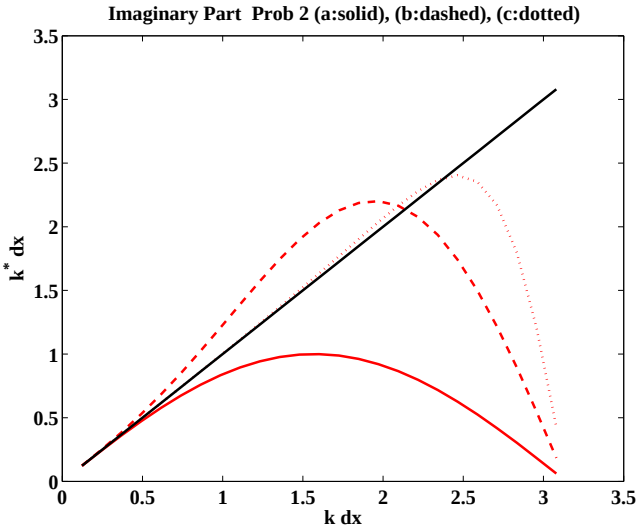
You can avoid doing all the complex algebra and just evaluate these terms in complex arithmetic for the next problem.

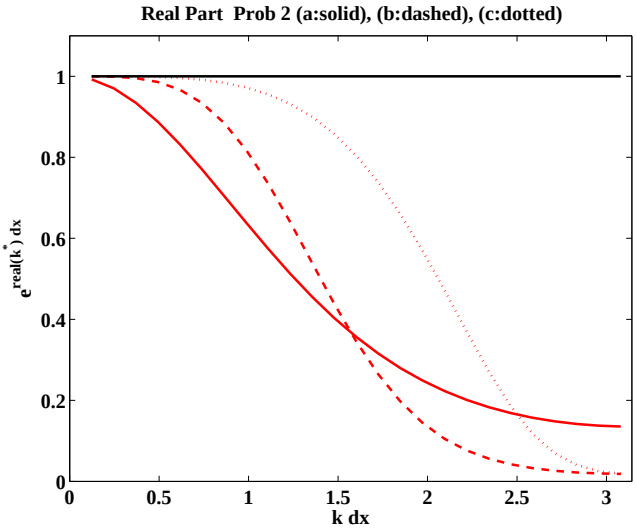
3. For problems 1 and 2 plot the resulting expressions for the $(k^*\Delta x)$ against $k\Delta x$ for $k = 1, 2, ...M/2$ with $M = 51$ and $\Delta x = 2\pi/M$. (You can use Matlab or anything else you want.)

Here's a sample Matlab code for my plots.

```
clear;

M = input('Enter M = ');

dx = 2.*pi/M;

k_dx = [1:M/2]*dx;

k_p1 = sin(k_dx);
k_p2 =4./3.*sin(k_dx) - 1./6.*sin(2.*k_dx);
k_p3 =3.*sin(k_dx)./(2.+cos(k_dx));

figure(1);
clf;

plot(k_dx,k_p1,'r',k_dx,k_p2,'r--',k_dx,k_p3,'r:',k_dx,k_dx,'k');
title('Modified Wave Number Prob 1 (a:solid), (b:dashed), (c:dotted)');
xlabel('k dx');
ylabel('k^* dx')_dx
k_p1 = (1.-cos(k_dx) + i*sin(k_dx));
k_p2 =(3. - 4.*(cos(k_dx) - i*sin(k_dx)) + ...
    (cos(2.*k_dx) - i*sin(2.*k_dx)))/2.0;
k_p3 =(2.-2.*cos(k_dx) + 3.*i*sin(k_dx))./(2.+cos(k_dx) - i*sin(k_dx));

figure(2);
clf;
plot(k_dx,imag(k_p1),'r',k_dx,imag(k_p2),'r--',k_dx,imag(k_p3),'r:',k_dx,k_dx,'k');
title('Imaginary Part  Prob 2 (a:solid), (b:dashed), (c:dotted)');
xlabel('k dx');
ylabel('k^* dx')

figure(3);
```








```
clf;
plot(k_dx,exp(-real(k_p1)),'r',k_dx,exp(-real(k_p2)),'r--',k_dx,exp(-real(k_p3)),'r:',...
    k_dx,ones(size(k_dx,2),1),'k');
axis([0 pi 0 1.1]);
title('Real Part  Prob 2 (a:solid), (b:dashed), (c:dotted)');
xlabel('k dx');
ylabel('e^{real(k^*) dx}')
```

(a) For problem 1 the results should all be pure imaginary, i.e. ik^*. Plot the imaginary part of $(ik^*\Delta x)$ against $k\Delta x$.

(b) For problem 2 the results should all be complex. Plot the imaginary part (a real number) against $k\Delta x$. Then plot exponential of (- real part $(ik^*)\Delta x$) against $k\Delta x$.