How wrong can a flow model be, and yet provide a reasonable acoustic prediction?

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A primitive approach to aeroacoustics...

- Geometry & flow conditions
- Transient CFD simulation
- Hydrodynamic near-field
- Aeroacoustic analogy
- Equivalent sources
- Numerical acoustics
- Acoustic far-field

sound generation
sound propagation
... and two questions

**First question**: why does a 2D model of periodic vortex pairing give linearly increasing pressure fluctuations?

\[
\begin{align*}
Z_L(t) &= u_0 t + \frac{d}{2} \cos \left( \frac{\Gamma t}{\pi d^2} \right) \\
R_L(t) &= R_0 + \frac{d}{2} \sin \left( \frac{\Gamma t}{\pi d^2} \right) \\
Z_T(t) &= u_0 t - \frac{d}{2} \cos \left( \frac{\Gamma t}{\pi d^2} \right) \\
R_T(t) &= R_0 - \frac{d}{2} \sin \left( \frac{\Gamma t}{\pi d^2} \right)
\end{align*}
\]

\[
p'(x, t) = \frac{\rho_0}{4c_0^2|\mathbf{x}|^3} \left\{ \left[ \left( -\frac{4\Gamma^4 R_0}{\pi^3 d^4} + \frac{3\Gamma^3 u_0}{\pi^2 d^2} \right) \cos \left( \frac{2\Gamma t}{\pi^2 d^2} \right) 
\right.ight.
\]
\[
- \left. \left. \frac{2\Gamma^4 u_0}{\pi^3 d^4} t \sin \left( \frac{2\Gamma t}{\pi^2 d^2} \right) \right] \mathbf{x} \cdot (\mathbf{n} \mathbf{n}) \cdot \mathbf{x}
\right.
\]
\[
\left. \left. + \left( \frac{2\Gamma^4 R_0}{\pi^3 d^4} - \frac{\Gamma^3 u_0}{\pi^2 d^2} \right) \cos \left( \frac{2\Gamma t}{\pi^2 d^2} \right) \left( \mathbf{x} \cdot \mathbf{x} \right) \right\} \right. \right. \]

???
... and two questions

Second question: why does Curle’s analogy always give me wrong results when I’m using an incompressible model for non-compact ducted flows?

\[ p_n(x, \omega) = -\frac{i}{2h} \sum_{n=0}^{\infty} \frac{\cos(\eta_n y)}{k_n C_n} \]

Exact solution (based on tailored Green’s function):

\[
\begin{align*}
&\left\{ k_n^2 \int_{S_0} \int \cos(\eta_n y_0) \rho_0 u^2 e^{\pm ik_n(x-x_0)} \, dx_0 \, dy_0 \\
&+ \eta_n^2 \int_{S_0} \int \cos(\eta_n y_0) \rho_0 v^2 e^{\pm ik_n(x-x_0)} \, dx_0 \, dy_0 \\
&\pm i k_n \eta_n \int_{S_0} \int \sin(\eta_n y_0) \rho_0 uv e^{\pm ik_n(x-x_0)} \, dx_0 \, dy_0 \right\}
\end{align*}
\]
Low Mach number issues

- Aeroacoustic analogies show a large sensitivity to seemingly small approximations in the flow model.
- Blame it on aeroacoustic analogies?
- At low Mach numbers: orders of magnitude of difference between hydrodynamic and acoustic fluctuations ($O(M^4)$)

\[
p' = 4.4934739 \text{ Pa}
\]

- Direct Noise Computations (DNC) → additional issues (dissipation/dispersion → numerical cost $\propto Re^2 M^{-4}$, NRBCs, synthetic turbulence generation, ...) 
- Are there better choices to be made in the decoupling between sound generation and propagation effects?
- Are there aeroacoustic analogies that perform better than others?
Plan

- Fundamentals of Lighthill’s acoustic analogy
  - Concepts
  - Approximations

- First question: sound emitted by free vortex leapfrogging
  - Review of Vortex Sound Theory
  - Predictions based on analytical model and experimental data

- Second question: sound emitted by ducted vortex leapfrogging, based on analytical model
  - Review of Curle’s analogy
  - The black magic behind the acoustic dipoles
Monopoles, dipoles, quadrupoles

- Monopole = pulsating sphere
  - Physically: unsteady combustion, pipe exhaust, vocal folds, ...

- Dipole = oscillating sphere without change of volume
  - Less efficient than monopole
  - Physically: unsteady forces

- Quadrupole = deforming sphere without change of volume nor net force
  - Less efficient than dipole
  - Physically: turbulence
Lighthill’s aeroacoustical analogy: concept

- The problem of sound produced by a turbulent flow is, **from the listener’s point of view**, analogous to a problem of propagation in a uniform medium at rest in which equivalent sources are placed.

- Wave propagation region: linear wave operator applies
  \[
  \frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \nabla^2 \rho' = q(\mathbf{x}, t)
  \]

- Turbulent region: fluid mechanics equations apply
  \[
  \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \\
  \rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \nabla \sigma + \mathbf{f}
  \]
  no external forces  
  \Rightarrow no dipoles
Lighthill’s analogy: definition of a reference state

- Reformulation of fluid mechanics equations, and use of arbitrary speed $c_0$:

$$\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \frac{\partial^2 \rho}{\partial x_i^2} = \frac{\partial^2 (\rho v_i v_j - \sigma_{ij})}{\partial x_i \partial x_j} + \frac{\partial^2 (p - c_0^2 \rho)}{\partial x_i^2}$$

- Definition of a reference state:

  $\rho' \equiv \rho - \rho_0$
  $p' \equiv p - p_0$
  $v_i' \equiv v_i$

- Aeroacoustical analogy:

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$$

  with $T_{ij} = \rho v_i v_j + (p' - c_0^2 \rho') \delta_{ij} - \sigma_{ij}$

  
  \textit{Lighthill’s tensor}

\textit{Exact... therefore perfectly useless!}
Sound produced by free isothermal turbulent flows at low Mach number: some useful approximations

- Solution using Green’s fct

\[
\rho'(x, t) = \int_{-\infty}^{t} \int_{V} \int_{V} \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \ G \ d^3 y \ d\tau - c_0^2 \int_{-\infty}^{t} \int_{\partial V} \left( \rho' \ \frac{\partial G}{\partial y_i} - G \ \frac{\partial \rho'}{\partial y_i} \right) n_i \ d^2 y \ d\tau
\]

Integral solution \(\rightarrow\) good for numerical stability

- Purpose: simplify the RHS

\[
T_{ij} = \rho v_i v_j + (p - \alpha v^2 \delta_{ij}) \delta_{ij} - \alpha_{ik} \]

\[
\rho v_i v_j \approx \rho_0 v_i v_j
\]

Explicit integral solution \(\rightarrow\) decoupling between source and propagation effects, allows using incompressible flow model
Lighthill’s $M^8$ law for low Mach number free jets

- Using free field Green’s function:
  $$\rho'(x, t) = \frac{\partial^2}{\partial x_i \partial x_j} \iint_V \left[ \frac{\rho_0 v_i v_j}{4\pi c_0^2 |x - y|} \right] \, d^3 y$$

- Scaling law:
  $$t^* = t - \frac{|x - y|}{c_0}$$
  
  Acoustic scale: \( x \propto \lambda = \frac{c_0}{f} \)
  Flow time scale: \( D/U_0 \)
  Spatial derivative: \( U_0/(c_0 D) \)

  \[
  p' = c_0^2 \rho' \propto \frac{U_0^2}{c_0^2 D^2} \frac{\rho_0 U_0^2 D^3}{|x|} \\
  = \rho_0 c_0^2 M^4 \frac{D}{|x|}
  \]

  Acoustical power:
  $$W = \frac{4\pi |x|^2 p'^2}{\rho_0 c_0} \propto \rho_0 c_0^3 D^2 M^8$$

**Quadrupolar source**

*Not by chance! We obtain quadrupolar radiation efficiency because we imposed it!*
**First question:** why does a 2D model of periodic vortex pairing give linearly increasing pressure fluctuations?

\[
p'(\mathbf{x}, t) = \frac{\rho_0}{4c_0^2|\mathbf{x}|^3} \left\{ \left[ \left( \frac{4\Gamma^4 R_0}{\pi^3 d^4} + \frac{3\Gamma^3 u_0}{\pi^2 d^2} \right) \cos \left( \frac{2\Gamma t}{\pi^2 d^2} \right) \right. \right. \\
&\left. \left. \quad - \frac{2\Gamma^4 u_0}{\pi^3 d^4} t \sin \left( \frac{2\Gamma t}{\pi^2 d^2} \right) \right] \mathbf{x} \cdot (\mathbf{n} \mathbf{n}) \cdot \mathbf{x} \right. \\
&\left. \quad + \left( \frac{2\Gamma^4 R_0}{\pi^3 d^4} - \frac{\Gamma^3 u_0}{\pi^2 d^2} \right) \cos \left( \frac{2\Gamma t}{\pi^2 d^2} \right) (\mathbf{x} \cdot \mathbf{x}) \right\}
\]
Vortex Sound Theory: Powell’s analogy

- Vectorial identity: \( \nabla \left( \frac{|v|^2}{2} \right) = v \times \omega + v \cdot \nabla v \)

- Momentum equation becomes:
  \[ \rho \frac{\partial v}{\partial t} + \rho \nabla \left( \frac{1}{2} |v|^2 \right) + \rho (\omega \times v) + \nabla p = 0 \]

- Similar manipulation as for Lighthill’s analogy:
  \[ \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} \frac{\partial^2 p'}{\partial x_i^2} = \nabla \cdot \left[ \rho (\omega \times v) + \nabla \left( \frac{1}{2} \rho |v|^2 \right) \right] \]

- Retaining leading order terms in \( M^2 \):
  \[ \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} \frac{\partial^2 p'}{\partial x_i^2} = \nabla \cdot \left[ \rho (\omega \times v) \right] + \nabla^2 \left( \frac{1}{2} \rho |v|^2 \right) \]

- Integral solution using free field Green’s function and first order Taylor expansion of the retarded time:
  \[ p'(x, t) = -\frac{x_i}{4\pi c_0 |x|^2} \frac{\partial}{\partial t} \iiint_V \rho (\omega \times v)_i \, d^3y - \frac{x_i x_j}{4\pi c_0^2 |x|^3} \frac{\partial^2}{\partial t^2} \iiint_V y_j \rho (\omega \times v)_i \, d^3y \]

- Powell’s integral formulation:
  \[ p_p'(x, t) = -\frac{\rho_0}{4\pi c_0^2 |x|^3} \frac{\partial^2}{\partial t^2} \iiint_V (x \cdot y) \, x \cdot (\omega \times v) \, d^3y \]
Starting from Powell’s integral formulation:

\[ p'_P(x, t) = -\frac{\rho_0}{4\pi c_0^2|x|^3} \frac{\partial^2}{\partial t^2} \iiint_V (x \cdot y) \cdot (\omega \times v) \, d^3y \]

Using vectorial’s identity:

\[ \nabla_y \times \left[ \frac{1}{3} (x \cdot y) x \times y \right] = (x \cdot y) x - \frac{1}{3} |x|^2 y \]

By substitution:

\[ p'(x, t) = -\frac{\rho_0}{12\pi c_0^2|x|^3} \frac{\partial^2}{\partial t^2} \iiint_V \nabla_y \times [(x \cdot y) x \times y] \cdot (\omega \times v) \, d^3y \]

\[ + |x|^2 \iiint_V y \cdot (\omega \times v) \, d^3y \]

Using Helmholtz’s vorticity transport equation:

\[ \frac{\partial \omega}{\partial t} + \nabla_y \times (\omega \times v) = 0 \]

Möhring’s integral formulation:

\[ p'_M(x, t) = \frac{\rho_0}{12\pi c_0^2|x|^3} \frac{\partial^3}{\partial t^3} \iiint_V (x \cdot y) x \cdot (y \times \omega) \, d^3y \]
Vortex Sound Theory: 
2 solutions for the same problem

- We have derived two (formally) equivalent formulations of the Vortex Sound Theory:
  - Powell’s analogy: 
    \[
    p_P^t(x, t) = -\frac{\rho_0}{4\pi c_0^2 |x|^3} \frac{\partial^2}{\partial t^2} \int \int \int_V (x \cdot y) x \cdot (\omega \times v) \, d^3 y
    \]
  - Mohring’s analogy: 
    \[
    p_M^t(x, t) = \frac{\rho_0}{12\pi c_0^2 |x|^3} \frac{\partial^3}{\partial t^3} \int \int \int_V (x \cdot y) x \cdot (y \times \omega) \, d^3 y
    \]

- Although formally equivalent, these two formulations do not yield the same numerical robustness!
Vortex Sound Theory for axisymmetrical flows

- Coordinate of a vortex element: \( y = z \mathbf{n} + r \mathbf{e}(\phi) \)

- General form of velocity and vorticity:
  \[
  \begin{cases}
  \omega(r, \phi, z) &= \omega(r, z) \mathbf{n} \times \mathbf{e}(\phi) \\
  \mathbf{v}(r, \phi, z) &= u(r, z) \mathbf{n} + v(r, z) \mathbf{e}(\phi) + w(r, z) \mathbf{n} \times \mathbf{e}(\phi)
  \end{cases}
  \]

- Powell’s analogy becomes:
  \[
  p'_p(x, t) = \frac{\rho_0}{4c_0^2|x|^3} \frac{\partial^2}{\partial t^2} \left\{ 2 \left( \int \int_S \omega vr z \, drdz \right) (x \cdot n)(x \cdot n) \ight.
  \]
  \[
  \quad - \left. \left( \int \int_S \omega v^2 r \, drdz \right) [(x \cdot x) - (x \cdot n)(x \cdot n)] \right\}
  \]

- Möhring’s analogy becomes:
  \[
  p'_M(x, t) = \frac{\rho_0}{4c_0^2|x|^3} \frac{d^3Q}{dt^3} x \cdot \left( n n - \frac{I}{3} \right) \cdot x \
  Q = \int \int_S \omega r^2 z \, drdz
  \]
Vortex ring pairing

- Vortex pairing = inviscid interaction (Biot-Savart)
  - Vortex leapfrogging: periodic motion
  - Vortex merging: requires core deformation

- Can be easily stabilized and studied at laboratory scale

- One of the mechanisms of sound production in subsonic jets

Sound prediction based on two different source models:

- Analytical models (2D and 3D axisymmetric)
  - controllable error

- Experimental data: time-resolved Particle Image Velocimetry
  - random error
2D and 3D models of vortex ring leapfrogging

- 2D model \((\sigma << d << R_0)\): locally planar interaction, neglects vortex stretching

\[
\begin{align*}
Z_L(t) &= u_0 t + \frac{d}{2} \cos \left( \frac{\Gamma t}{\pi d^2} \right) \\
R_L(t) &= R_0 + \frac{d}{2} \sin \left( \frac{\Gamma t}{\pi d^2} \right) \\
Z_T(t) &= u_0 t - \frac{d}{2} \cos \left( \frac{\Gamma t}{\pi d^2} \right) \\
R_T(t) &= R_0 - \frac{d}{2} \sin \left( \frac{\Gamma t}{\pi d^2} \right)
\end{align*}
\]

- 3D model \((\sigma << d = O(R_0))\): accounts for vortex stretching

\[
\begin{align*}
\frac{dZ_m}{dt} &= \frac{1}{R_m} \frac{\partial \Psi}{\partial R_m} + \frac{\Gamma_m}{4\pi R_m} \left[ \log \left( \frac{8R_m}{\sigma_{c,m}} \right) - \frac{1}{4} \right] \\
\frac{dR_m}{dt} &= -\frac{1}{R_m} \frac{\partial \Psi}{\partial Z_m} \\
\frac{d\sigma^2_{c,m} R_m}{dt} &= 0
\end{align*}
\]

\[
\Psi = \frac{\Gamma_n}{2\pi} \sqrt{R_m R_n} \left[ \left( \frac{2}{k_{mn} - k_{imn}} \right) K(k_{mn}) - \frac{2}{k_{mn}} E(k_{mn}) \right]
\]

\[
k_{mn} = \sqrt{\frac{4R_m R_n}{(Z_m - Z_n)^2 + (R_m + R_n)^2}}
\]
2D model: vortex trajectories and flow invariants

- Two cases considered: $d / R_0 = 0.1$ and $0.3$
- Locus of the vortex cores:

![Diagram showing vortex trajectories and flow invariants](image)

- Flow invariants:

\[
P = 2\pi \rho_0 \Gamma R_0^2 \left[ 1 + \frac{d^2}{4R_0^2} \sin^2 \left( \frac{\Gamma t}{\pi d^2} \right) \right]
\]

\[
T = 2\pi \rho_0 \Gamma \left[ 2R_0^2 u_0 - \frac{\Gamma R_0}{2\pi} + \left( u_0 \frac{d^2}{2} - \frac{\Gamma R_0}{2\pi} \right) \sin^2 \left( \frac{\Gamma t}{\pi d^2} \right) - \frac{u_0 \Gamma}{4\pi} t \sin \left( \frac{2\Gamma t}{\pi d^2} \right) \right]
\]

secular term
2D model: sound prediction

- Powell’s analogy:

\[ p'_P(x, t) = \frac{\rho_0}{4c_0^2|x|^3} \left\{ \left[ \left( -\frac{4\Gamma^4R_0}{\pi^3d^4} + \frac{3\Gamma^4u_0}{\pi^2d^2} \right) \cos \left( \frac{2\Gamma t}{\pi^2 d^2} \right) \right. \right. \]
\[ - \left. \left. \frac{2\Gamma^4u_0}{\pi^3d^4} t \sin \left( \frac{2\Gamma t}{\pi^2 d^2} \right) \right] x \cdot (n \cdot n) \cdot x \right. \]
\[ + \left. \left( \frac{2\Gamma^4R_0}{\pi^3d^4} - \frac{\Gamma^2u_0^2}{\pi^2d^2} \right) \cos \left( \frac{2\Gamma t}{\pi^2 d^2} \right) (x \cdot x) \right\} \]

- Möhring’s analogy:

\[ p'_M(x, t) = \frac{\rho_0}{4c_0^2|x|^3} \left\{ \left( \frac{3\Gamma^4u_0}{\pi^2d^2} - \frac{4\Gamma^4R_0}{\pi^3d^4} \right) \cos \left( \frac{2\Gamma t}{\pi^2 d^2} \right) \right. \]
\[ - \left. \frac{2\Gamma^4u_0}{\pi^3d^4} t \sin \left( \frac{2\Gamma t}{\pi^2 d^2} \right) \right] x \cdot (n \cdot n - \frac{I}{3}) \cdot x \]

- Conclusion: failure of both Powell’s and Möhring’s analogies when applied to a flow model that does not respect the conservation of momentum and kinetic energy
Möhring’s solution (trick?): reinforcement of physical assumptions

\[ p'_M(x, t) = \frac{\rho_0}{4c_0^2|x|^3} \frac{d^3Q}{dt^3} x \cdot \left( nn - \frac{I}{3} \right) \cdot x \]

- Using Lamb (1932) identities:
  \[ \frac{dQ}{dt} = \frac{T}{2\pi\rho_0} + 3\Gamma (R_L V_L Z_L + R_T V_T Z_T) \]

- Imposing further conservation of momentum:
  \[ \frac{d}{dt} (R_L^2 + R_T^2) = 0 \quad R_L V_L = -R_T V_T \]

- We obtain:
  \[ \frac{dQ}{dt} = \frac{T}{2\pi\rho_0} + 3\Gamma R_L V_L (Z_L - Z_T) \]

- Imposing further conservation of kinetic energy:
  \[ p'(x, t) = -\frac{3}{4} \frac{\rho_0 \Gamma^4 R_0}{\pi^3 d^4 c_0^2 |x|^3} \cos \left( \frac{2\Gamma t}{\pi d^2} \right) x \cdot \left( nn - \frac{I}{3} \right) \cdot x \]
Experimental facility

- Two flow conditions:
  - $U_0 = 5 \text{ m/s (Re = 14,000 $\rightarrow$ stable laminar vortices but negative decibels...)}$
  - $U_0 = 34 \text{ m/s (Re = 93,000 $\rightarrow$ more turbulent vortices but measurable SPL)}$
Application to PIV data
PIV results: acoustical source terms

\[ \frac{T}{2\pi\rho_0} + 3 \int_S \omega v r (z - z_0) \, dr \, dz \]

\[ = \frac{\int_S \omega r^2 \, dr \, dz}{\int_S \omega r^2 \, dr} \]

\[ - 3 \int_S \omega v r (z - z_0) \, dr \, dz \]
Acoustic predictions: PIV vs. tuned analytical models

5.0 m/s

34.2 m/s
Acoustic predictions: PIV vs. acoustic measurements
**First question:** why does a 2D model of periodic vortex pairing give linearly increasing pressure fluctuations?

**Answer:** that’s because I had let my acoustic analogy believe that my wrong flow model was... wrong!

\[ p'(x, t) = \frac{\rho_0}{4c_0^2|x|^3} \left\{ \left( -\frac{4\Gamma^4 R_0}{\pi^3 d^4} + \frac{3\Gamma^3 u_0}{\pi^2 d^2} \right) \cos \left( \frac{2\Gamma t}{\pi^2 d^2} \right) - \frac{2\Gamma^4 u_0}{\pi^3 d^4} t \sin \left( \frac{2\Gamma t}{\pi^2 d^2} \right) \right\} x \cdot (nn) \cdot x + \left( \frac{2\Gamma^4 R_0}{\pi^3 d^4} - \frac{\Gamma^3 u_0}{\pi^2 d^2} \right) \cos \left( \frac{2\Gamma t}{\pi^2 d^2} \right) (x \cdot x) \]

\[ p'(x, t) = -\frac{3}{4} \frac{\rho_0 \Gamma^4 R_0}{\pi^3 d^4 c_0^2 |x|^3} \cos \left( \frac{2\Gamma t}{\pi d^2} \right) \left( \textbf{x} \cdot \left( \textbf{nn} - \frac{\textbf{I}}{3} \right) \right) \cdot \textbf{x} \]
**Second question:** why does Curle’s analogy always give me wrong results when I’m using an incompressible model for non-compact ducted flows?

\[
p_a(x, \omega) = -\frac{i}{2h} \sum_{n=0}^{\infty} \frac{\cos(\eta_n y)}{k_n C_n}
\]

**Exact solution (based on tailored Green’s function):**

\[
\begin{align*}
&\left\{ \frac{k_n^2}{2} \int_{S_0} \cos(\eta_n y_0) \rho_0 u^2 e^{\pm ik_n(x-x_0)} \, dx_0 \, dy_0 \\
&+ \frac{\eta_n^2}{2} \int_{S_0} \cos(\eta_n y_0) \rho_0 v^2 e^{\pm ik_n(x-x_0)} \, dx_0 \, dy_0 \\
&\pm i k_n \eta_n \int_{S_0} \sin(\eta_n y_0) \rho_0 u v e^{\pm ik_n(x-x_0)} \, dx_0 \, dy_0 \right\}
\end{align*}
\]

**Exact:**

**Curle:**
Curle’s analogy: fixed rigid bodies

- Lighthill’s aeroacoustical analogy:
  \[
  \frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}
  \]

- Integral solution using Green’s function
  \[
  \rho'(x, t) = \int_{-\infty}^{t} \int_{V} \int_{\partial V} \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} G \, d^3 y \, d\tau \quad \text{incident field}
  \]
  \[
  - c_0^2 \int_{-\infty}^{t} \int_{\partial V} \left( \rho' \frac{\partial G}{\partial y_i} - G \frac{\partial \rho'}{\partial y_i} \right) n_i \, d^2 y \, d\tau \quad \text{scattered field}
  \]

- Partial integration of source integral
  \[
  \int_{-\infty}^{t} \int_{V} \int_{\partial V} \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} G \, d^3 y \, d\tau = \int_{-\infty}^{t} \int_{V} \int_{\partial V} T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} \, d^3 y \, d\tau
  \]
  \[
  + \int_{-\infty}^{t} \int_{\partial V} \left\{ \left( - \frac{\partial \rho_i}{\partial \tau} - c_0^2 \frac{\partial \rho'}{\partial y_i} \right) G - \left( \rho_i v_j + (\rho' - c_0^2 \rho') \delta_{ij} + \sigma_{ij} \right) \frac{\partial G}{\partial y_j} \right\} n_i \, d^2 y \, d\tau
  \]

- Curle’s analogy: uses free field Green’s function
  \[
  G_0(t, x|\tau, y) = \frac{\delta(t - \tau - |x - y|/c_0)}{4\pi c_0^2 |x - y|}
  \]

\[
\rho'(x, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int_{V} \int_{\partial V} \left[ \frac{T_{ij}}{4\pi c_0^2 |x - y|} \right] d^3 y - \frac{\partial}{\partial x_i} \int_{\partial V} \left[ \frac{\rho' n_i}{4\pi c_0^2 |x - y|} \right] d^2 y
\]

**Quadrupole, W \propto M^8**
**in free field**

**Dipole, W \propto M^6**
**in free field**
The black magic behind Curle’s acoustic dipoles

- No assumption made in Curle’s analogy, only a reformulation of Navier-Stokes equations solved using Green’s function
- But hard-wall boundary condition has disappeared, replaced by equivalent dipoles distributed over solid surfaces

→ The dipoles must represent the non-penetration boundary condition, for
  - the hydrodynamic problem: reaction force exerted by the walls subjected to wall-normal flow momentum, and
  - the acoustic problem: scattering of incident acoustic waves.

- But what if my dipoles are computed from an incompressible flow model?
Second question: why does Curle’s analogy always give me wrong results when I’m using an incompressible model for non-compact ducted flows?

Answer: that’s because the dipoles lack the compressible component corresponding to acoustic scattering, leading to acoustic leakage effects.

Does it imply incompressible models cannot be used for such cases? No, but the distinction between hydrodynamic and acoustic effects must be introduced explicitly.
Boundary Integral discretization: bringing the listener inside the source region

- **Direct Boundary Element Method (DBEM)**
  - Derivation essentially similar to Curle’s analogy
    - Resolution of the Helmholtz equation \( \nabla^2 \hat{p}_a + k^2 \hat{p}_a = q_L \)
  
  with the source term
  \[
  \hat{q}_L = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(- \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}\right) e^{-i\omega t} \, dt = -\frac{\partial^2 \hat{T}_{ij}}{\partial x_i \partial x_j}
  \]
  
  using free field Green’s function
  \[
  G = \frac{e^{-ikr}}{4\pi r}, \quad r = |x - y| \quad \nabla^2 G + k^2 G = -\delta(x - y)
  \]

- **Collocation method**: the integral solution is evaluated over the acoustic mesh
  - Exclude Green’s kernel singularity!

  \[
  \int_{V \setminus V_c} \left( \nabla^2 p_a \, G - p_a \, \nabla^2 G \right) \, d^3y = \int_{V \setminus V_c} q_L \, G \, d^3y + \int_{V \setminus V_c} p_a \, \delta(x - y) \, d^3y
  \]

  \[
  C'(x) \, p_L(x) = \int \int \int_V T_{ij} \left( \frac{\partial^2 G}{\partial y_i \partial y_j} \right) \, d^3y - \int \int_{\partial V} p_L \left( \frac{\partial G}{\partial n} \right) \, d^2y
  \]
Contributions of compact and non-compact regions

- We decompose the pressure into acoustic and hydrodynamic: \( p_L = p_h + p_a \)

\[
C(x) p_a(x) = - \iint_{\partial V} p_a \frac{\partial G}{\partial n} \, d^2 y + \iiint_{V_2} T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} \, d^3 y - \iint_{\partial V_2} p_h \frac{\partial G}{\partial n} \, d^2 y
\]

- The near-field effects are of hydrodynamic nature and taken care of by the incompressible flow solver
- The far-field effects are of compressible nature and obtained through the BEM solver
Validation: spinning vortex pair in straight duct

- **Purpose:** perform unambiguous validation of the BEM / Curle approach
  - Flow should be amenable to (nearly) exact modelling
  - Geometry should allow an exact evaluation of the scattering (tailored Green’s fct)
  - Incompressible flow model

- **Leapfrogging of 2 rectilinear vortex filaments in an infinite 2D duct**
  - Flow kinematics: based on the complex potentials of the system of 2 vortices, and of the infinite series of image vortices
  - Non-penetration, slip condition at both walls (streamlines)
Flow model: source fields and vortex desingularization

- Volume source field:
  1. Compute the vortex kinematics by time marching the equations

\[ u_m = -\frac{\Gamma}{4h} \left\{ \frac{\sin \left[ \pi \left( y_m - y_n \right) / h \right]}{\cosh \left[ \pi \left( x_m - x_n \right) / h \right] - \cos \left[ \pi \left( y_m - y_n \right) / h \right]} + \frac{\sin \left[ \pi \left( y_m + y_n \right) / h \right]}{\cosh \left[ \pi \left( x_m - x_n \right) / h \right] + \cos \left[ \pi \left( y_m + y_n \right) / h \right]} \right\} + \frac{\sin \left( 2\pi y_m / h \right)}{1 + \cos \left( 2\pi y_m / h \right)} \]

\[ v_m = -\frac{\Gamma}{4h} \left\{ \frac{-\sinh \left[ \pi \left( x_m - x_n \right) / h \right]}{\cosh \left[ \pi \left( x_m - x_n \right) / h \right] - \cos \left[ \pi \left( y_m - y_n \right) / h \right]} + \frac{\sinh \left[ \pi \left( x_m - x_n \right) / h \right]}{\cosh \left[ \pi \left( x_m - x_n \right) / h \right] + \cos \left[ \pi \left( y_m + y_n \right) / h \right]} \right\} \]

2. Compute the induced velocity field using the desingularized kernel

\[ v_\theta(r) = \frac{\Gamma}{2\pi r} \left[ 1 - \exp \left( - \frac{r^2}{2\sigma^2} \right) \right] \]

- Wall pressure: integrated from unsteady Bernoulli’s eq.: 
  \[ p_w = -\rho \left( \frac{\partial \Phi_w}{\partial t} + \frac{u_w^2}{2} \right) \]
Source fields: $T_{xy}$ and wall pressure
Reference solution used for validation

- Reference solution: based on tailored Green’s function

\[ G_1 = \frac{i}{2h} \sum_{n=0}^{\infty} \frac{1}{C_n k_n} \cos(\eta_n y_0) \cos(\eta_n y) e^{\pm ik_n(x-x_0)} \]

\[ C_n = \begin{cases} 
1 & \text{if } n = 0 \\
1/2 & \text{if } n \neq 0 
\end{cases} \]

\[ p_n(x, \omega) = -\frac{i}{2h} \sum_{n=0}^{\infty} \frac{\cos(\eta_n y)}{k_n C_n} \left\{ k_n^2 \iint_{S_0} \cos(\eta_n y_0) \rho_0 u^2 e^{\pm ik_n(x-x_0)} \, dx_0 \, dy_0 \right. \]

\[ + \eta_n^2 \iint_{S_0} \cos(\eta_n y_0) \rho_0 v^2 e^{\pm ik_n(x-x_0)} \, dx_0 \, dy_0 \]

\[ \pm ik_n \eta_n \iint_{S_0} \sin(\eta_n y_0) \rho_0 uv e^{\pm ik_n(x-x_0)} \, dx_0 \, dy_0 \right\} \]
Validation of the Curle/DBEM method for $kh = 4.8$

Reference solution

Curle/DBEM

Dipolar contribution

Quadrupolar contribution

Lighthill
Summary

- Aeroacoustical analogies allow extracting a maximum of acoustical information from a given description of the flow field.
- In absence of flow-acoustic coupling, an incompressible flow model permits to obtain a reasonable acoustic prediction.

- Numerical robustness issues have been reviewed:
  - In cases of non-compact source regions, the incompressible solution must be complemented by an acoustic correction to account for scattering effects.
  - The prediction of free jet quadrupolar noise requires a flow model free of spurious monopolar or dipolar sources, which can be otherwise explicitly discarded using an appropriate source formulation.

- Without approximations, the analogy is useless!
Thanks for your attention!

Questions?